The Dissertation Committee for Arun Albert Debray certifies that this is the approved version of the following dissertation:

Taking topological field theory at phase value

Committee:

Dan Freed, Supervisor

David Ben-Zvi

Jacques Distler

Andrew Neitzke

Sam Raskin

Taking topological field theory at phase value

by

Arun Albert Debray

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Taking topological field theory at phase value

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Arun Albert Debray, PhD
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SUPERVISOR: Dan Freed

In this thesis, we use methods of topological field theory to model and study topological phases of matter. This includes computing TFTs that capture low-energy information for the GDS model and for the Majorana chain with time-reversal symmetry. We then investigate phases of matter with spatial symmetries that mix with the internal symmetry type; we provide a mathematical model for these phases and prove a "fermionic crystalline equivalence principle" theorem as predicted in the physics literature. Some of our computations lead to a bonus theorem on the classification of some unorientable 4-manifolds up to stable diffeomorphism.

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CHAPTER 1

Introduction

"I am hitting my head against the walls, but the walls are giving way." – attributed to Gustav Mahler

1.0. Overview and summary of results

This thesis presents research mathematically modeling certain physical systems. Specifically, we use methods of algebraic topology and topological field theory to study topological phases of matter. This work is part of a broader research context in mathematics and theoretical physics in the last decade making progress on formalizing models for topological phases, classifying the different phases that can occur, and studying their physical properties; despite significant progress in our understanding of topological phases, several open questions remain at the root of the field. Notably, there is not yet a general definition for a topological phase of matter. Current classification methods make use of examples or heuristic definitions of topological phases, and often rely on further heuristics to define invariants used in their classification results.

We do not solve these problems in this thesis. What we do instead is study examples or classes of examples of topological phases, modeling these phases and using these models to gain insights about the classification conjectures.

- One approach towards classifying topological phases is to try to extract an invariant called the *low-energy TFT*, which is conjectured to be a complete invariant. In Chapter 2, we study this idea for the *generalized double semion (GDS)* model, whose low-energy TFT was not previously known in general. Though it is not yet possible to determine the entire low-energy TFT, we define a TFT Z_{GDS} and prove Theorem 2.3.19, a result suggesting that Z_{GDS} is the low-energy TFT of the GDS model. The content of this chapter is published as [**Deb20**].
- In Chapter 3, joint with Sam Gunningham, we study the low-energy behavior of a different model, the Majorana chain. The phase predicted to be associated to this system is an example of a special class of phases called *symmetry-protected topological (SPT)* phases, which are conjectured to correspond to invertible TFTs in the low-energy ansatz. Specifically, it is believed that the group of 2d fermionic SPT phases with a time-reversal symmetry squaring to 1 is isomorphic to $\mathbb{Z}/8$, and

that the phase of the Majorana chain is a generator. In the low-energy ansatz, this is related to the $\mathbb{Z}/8$ classification of 2d pin$^-$ reflection positive invertible TFTs, generated by the *Arf-Brown TFT* Z_{AB}. We give a few different constructions of the Arf-Brown invariant, which is the partition function of Z_{AB}, in §3.2, then construct Z_{AB} in §3.4. In §3.5, we discuss the Majorana chain, and in Corollary 3.5.30, we prove a result suggesting that Z_{AB} or an odd multiple thereof is the low-energy TFT of the Majorana chain. The content of this chapter is published as [**DG18**].

- In Chapter 4, we study *crystalline SPT phases*, which are SPT phases for which the symmetry group G acts on space. Building on an ansatz of Freed-Hopkins, we model these phases using *phase homology groups*, defined using Borel-equivariant parametrized homotopy theory. We then prove a "fermionic crystalline equivalence principle" (Theorem 4.2.8) calculating phase homology groups in terms of groups of reflection positive invertible field theories. Using this, in §4.4 and §4.5 we calculate phase homology groups providing models for classifications of various classes of crystalline SPT phases, finding agreement with the literature where these classes were already considered and offering predictions for the remaining classes. The content of this chapter is posted as a preprint [**Deb21a**] on the ArXiv.
- The last chapter, Chapter 5, tackles a question in topology, using similar computations as in Chapter 4 to a different end, classifying 4-manifolds up to stable diffeomorphism. We prove that for some classes of unorientable 4-manifolds, the classification simplifies, and give the complete classification in those cases in Theorems 5.3.2, 5.3.5, 5.4.2 and 5.4.5. The content of this chapter is posted as a preprint [**Deb21b**] on the ArXiv.

The rest of the current chapter contains more background and detail. §1.1 introduces bordism and the homotopy-theoretic preliminaries we use in the rest of the thesis; §1.2 introduces topological field theory; §1.3 goes over Freed-Hopkins-Teleman's homotopy-theoretic classification of invertible TFTs; and §1.4 discusses topological phases of matter, their models using lattice Hamiltonians, and the low-energy classification ansatz. Finally, in §1.5, we summarize the main results of the thesis.

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1.1. Bordism and other homotopy-theoretic preliminaries

Bordism is a tool in algebraic topology building algebraic data out of the geometric question of which manifolds are boundaries of other manifolds. Mostly it appears in this thesis thanks to results of Freed-Hopkins-Teleman [FHT10] and Freed-Hopkins [FH16a, Theorem 1.1] classifying different kinds of invertible topological field theories using bordism. Bordism plays an important direct role in Chapters 3, 4 and 5, and a more background or implicit role in Chapter 2.

In this section, we introduce bordism (§1.1.1); define Thom spectra and relate them to bordism (§1.1.2); discuss the Thom isomorphism and orientations (§1.1.3); discuss some tools for computing bordism groups (§1.1.5 and §1.1.6) and define the Anderson and Brown-Comenetz duals of the sphere (§1.1.7), which along with Thom spectra will play a role in the classification of invertible field theories, as we discuss in §1.3.

This section, like the rest of the thesis, assumes some familiarity with spectra in the sense of stable homotopy theory. The necessary background can be read in [**FH16a**, §6.1]. In Chapter 4 we require a little more background, and we provide references there.[1]

1.1.1. Bordism groups. Let $B\mathrm{O}$ denote the classifying space of the infinite-dimensional orthogonal group $\mathrm{O} \coloneqq \varinjlim_n \mathrm{O}_n$. Homotopy classes of maps $X \to B\mathrm{O}$ are in natural bijection with isomorphism classes of stable virtual vector bundles on X. If M is a smooth manifold, the tangent bundle defines a map $TM\colon M \to B\mathrm{O}$. There is another map $\nu\colon M \to B\mathrm{O}$, called the *stable normal bundle*, defined as follows: by results of Whitney [**Whi44**] and Wu [**Wu58**], for $N \gg 0$, there is a unique isotopy class of embeddings $M \hookrightarrow S^N$, hence a unique isomorphism class of normal bundle for said embedings. Including $S^N \hookrightarrow S^{N+1}$ adds on a trivial summand to the normal bundle, so the stable isomorphism type of the normal bundle is independent of N. We let ν be a representative of the homotopy class of maps classifying this isomorphism type.

Definition 1.1.1. A *symmetry type* is a fibration $\xi\colon B \to B\mathrm{O}$.[2]

Let M be a smooth manifold.

- A *(tangental) ξ-structure* on M is a homotopy class of a lift

$$\begin{tikzcd} & B \arrow[d, "\xi"] \\ M \arrow[ur, dashed] \arrow[r, "TM"'] & B\mathrm{O}. \end{tikzcd} \tag{1.1.2}$$

- A *normal ξ-structure* on M is the same thing, except with ν in place of TM.
- More generally, a ξ-structure on a vector bundle $E \to X$, classified by a map $E\colon X \to B\mathrm{O}$, is a lift of that map across ξ.

[1]For a more in-depth introduction to spectra, see Schwede's book [**Sch**]. The construction of well-behaved categories of spectra was a major undertaking in homotopy theory, with important steps taken by Lima [**Lim59, Lim60**], Boardman [**Boa65**], Adams [**Ada74**, Part III], Elmendorf-Kriz-Mandell-May [**EKMM97**], Mandell-May-Schwede-Shipley [**MMSS01**], and Lurie [**Lur17**, §1.4.3].

[2]Up to homotopy equivalence, any map can be made a fibration, so the condition that ξ be a fibration is not really a restriction.

Example 1.1.3 (G-structures)**.** Let G_n be a Lie group and $\rho\colon G_n \to \mathrm{O}_n$ be a representation. Under some conditions discussed by Freed-Hopkins [**FH16a**, §2.1], this data stabilizes to define a symmetry type $\xi\colon BG \to B\mathrm{O}$. Important examples include:

(1) $\mathrm{id}\colon \mathrm{O}_n \to \mathrm{O}_n$ stabilizing to $\mathrm{id}\colon B\mathrm{O} \to B\mathrm{O}$. This structure on a vector bundle is no additional data.
(2) $\mathrm{SO}_n \hookrightarrow \mathrm{O}_n$ stabilizes to $B\mathrm{SO} \to B\mathrm{O}$. This structure is equivalent to an orientation on a vector bundle.
(3) For $n \geq 3$, $\mathrm{Spin}_n \twoheadrightarrow \mathrm{SO}_n$ can be defined as the universal cover, which defines Spin_n and its map to SO_n up to isomorphism of this data. Composing with the inclusion $\mathrm{SO}_n \hookrightarrow \mathrm{O}_n$ and stabilizing defines a symmetry type $B\mathrm{Spin} \to B\mathrm{O}$ corresponding to spin structures.
(4) Since $\pi_1\mathrm{O}_n \cong \mathbb{Z}/2$ for $n \geq 3$, there are two different universal covering groups $\mathrm{Pin}_n^\pm \twoheadrightarrow \mathrm{O}_n$. Along the same line of reasoning we obtain $B\mathrm{Pin}^\pm \to B\mathrm{O}$ and pin$^+$ and pin$^-$ structures on vector bundles. Pin$^-$ structures will play an important role in Chapter 3.

In these examples, a G_n-structure on a vector bundle is equivalent data to a reduction of structure group of the frame bundle of a vector bundle from O_n to G_n.

Remark 1.1.4. For us, symmetry types often arise as encoding the (topological) information needed to define a field theory. For example, a theory with spinors has to be formulated on manifolds with spin structure, or perhaps a variant thereof. A theory with time-reversal symmetry can be put on unorieted manifolds. Determining the symmetry type is an important first step in formulating a mathematical question about field theory.

In Chapter 5, we will use symmetry types in a different way, as encoding low-degree homotopical information in a manifold.

Suppose M is a manifold with boundary. The normal bundle to $\partial M \hookrightarrow M$ is trivial, and has two homotopy classes of trivializations $\nu \cong \underline{\mathbb{R}}$. The *inward normal* is the class of trivializations in which the section $x \mapsto 1 \in \mathbb{R} = \underline{\mathbb{R}}_x$ is in the direction of M; the *outward normal* is the other class of trivializations.

Given a trivialization of ν, we get a stable isomorphism $T(\partial M) \simeq TM|_{\partial M}$, and therefore a ξ-structure on M induces a ξ-structure on ∂M. The two trivializations of ν give us two induced ξ-structures; we denote by ∂M the ξ-structure induced from the inward normal trivialization, and $-\partial M$ the ξ-structure induced from the outward normal trivialization. In some cases these ξ-structures coincide, but not always: for example, for orientations, these two induced structures are oppositely oriented.

Definition 1.1.5. Fix a symmetry type $\xi\colon B \to B\mathrm{O}$. We define an equivalence relation on the set[3] of closed, n-dimensional ξ-manifolds by saying that $M \sim N$ if there is a compact $(n+1)$-dimensional ξ-manifold and a diffeomorphism $\partial X \xrightarrow{\cong} M \amalg N$ inducing an equivalence of ξ-structures $\partial X \cong M \amalg (-N)$. Equivalent manifolds are called *bordant*, and X is called a *bordism* from M to N. The set of equivalence classes is denoted Ω_n^ξ.

Is this in fact an equivalence relation? Reflexivity and symmetry are obvious; transitivity uses the fact that bordisms glue, as depicted in Figure 1.

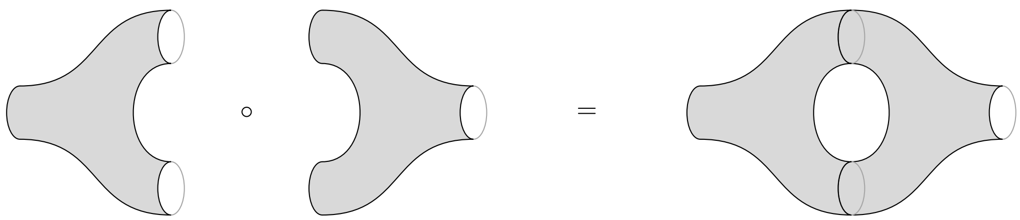

FIGURE 1. Gluing (composing) bordisms.

Lemma 1.1.6. *Ω_n^ξ is an abelian group under disjoint union, with the empty set (with its unique ξ-structure) as the identity.*

PROOF. Given a ξ-manifold M, we want to construct an inverse M^{-1}, meaning $M \amalg M^{-1}$ is the boundary of some compact n-dimensional ξ-manifold. The cylinder $M \times [0,1]$ is a bordism from M to itself, meaning $\partial(M \times [0,1]) \cong M \amalg (-M)$, as desired: $M^{-1} = -M$. $\square$

Computing these bordism groups for various choices of ξ was an important classical problem in algebraic topology.

If ξ is a G-structure as in Example 1.1.3, we often denote Ω_n^ξ as Ω_n^G, e.g. Ω_*^{SO}, Ω_*^{Spin}, and so on.

Remark 1.1.7. For some symmetry types ξ, Ω_*^ξ is a ring under direct product of ξ-manifolds. For example, the product of two oriented manifolds is canonically oriented. This applies to $\xi = \mathrm{O}$, SO, Spin, Spin^c, and String,[4] but not to many other bordism theories, such as pin$^\pm$ or pin^c bordism.

Classically, determining these ring structures was an important (and hard!) question, but in applications of bordism to physics, this structure tends to be less important.

[3] The size issue suggested by this definition can be finessed by fixing an infinite-dimensional vector space V and studying submanifolds of V.

[4] A spin manifold M has a characteristic class $\lambda \in H^4(M;\mathbb{Z})$ such that $2\lambda = p_1$. A *string structure* is a trivialization of λ [Gia71].

One's first instinct may be to calculate bordism groups using methods in geometric topology: for example, the classification of closed, oriented surfaces implies $\Omega_2^{\mathrm{SO}} = 0$, and the more intricate arguments of Rohlin [**Roh51**], Thom [**Tho52**], Lickorisch [**Lic63**], Kaplan [**Kap79**], Rourke [**Rou85**], Melvin-Kazez [**MK89**], Ancel-Guilbault [**AG92**], Casali-Gagliardi [**CG97**], Stipsicz [**Sti00**], and Constantino-Thurston [**CT08**] show $\Omega_3^{\mathrm{O}} = 0$, $\Omega_3^{\mathrm{SO}} = 0$, and $\Omega_3^{\mathrm{Spin}} = 0$. However, these techniques grow difficult quickly in n, and most computations of bordism groups use more homotopical methods.

1.1.2. Thom spaces and Thom spectra.

Definition 1.1.8. Let $V \to X$ be a vector bundle. The *Thom space* $\mathrm{Th}(X, V)$ is the quotient $D(V)/S(V)$, where $D(V)$ is the unit disc bundle of V and $S(V)$ is the unit sphere bundle. This is a pointed space with basepoint the class of $S(V)$.

To make such a definition, we must choose a Euclidean metric; all such choices yield homeomorphic Thom spaces, so we abuse notation and refer to "the" Thom space. There is a homeomorphism

$$\mathrm{Th}(X, \underline{\mathbb{R}}^n) \cong \Sigma^n X, \tag{1.1.9}$$

and more generally $\mathrm{Th}(X, V \oplus \underline{\mathbb{R}}) \cong \Sigma\mathrm{Th}(X, V)$, so the Thom space may be interpreted as a twisted suspension. A map $f\colon Y \to X$ induces a map $f_*\colon \mathrm{Th}(Y, f^*V) \to \mathrm{Th}(X, V)$.[5]

Recall that a *virtual vector bundle* on a space X is a formal difference of (real) vector bundles, with an identification $E - F = (E \oplus G) - (F \oplus G)$. Recall $B\mathrm{O}$ is the classifying space for virtual stable vector bundles; therefore by adding or subtracting trivial summands, we can regard $B\mathrm{O}$ as the classifying space for rank-zero virtual vector bundles. $B\mathrm{O}$ is an H-group under direct sum (i.e. like a topological group, but up to homotopy).

Definition 1.1.10. Let $V \to X$ be a virtual vector bundle and let $\xi\colon B \to B\mathrm{O}$ denote (a representative of) the classifying map for V; the homotopy class of ξ is uniquely defined. For $n \in \mathbb{N}$, let $B_n \coloneqq B\mathrm{O}_n \times_{B\mathrm{O}} B$ and $V_n \to B_n$ be the pullback of V along $B_n \to B$. Then the pullback of V_{n+1} along $b_n\colon B_n \to B_{n+1}$ is isomorphic to $V_n \oplus \underline{\mathbb{R}}$.

The *Thom spectrum* of V, denoted B^V, is the spectrum whose n^{th} space is $\mathrm{Th}(B_n, V_n)$, and whose structure map is

$$\Sigma\mathrm{Th}(B_n, V_n) \cong \mathrm{Th}(B_n, V_n \oplus \underline{\mathbb{R}}) \cong \mathrm{Th}(B_n, b_n^* V_{n+1}) \xrightarrow{(b_n)_*} \mathrm{Th}(B_{n+1}, V_{n+1}). \tag{1.1.11}$$

[5]To define this map, we need to choose a Euclidean metric on V and use the pullback metric on f^*V; the homotopy class of this map does not depend on this choice. There are a few other constructions in this subsection which need a choice of metric to define in a similar way, and which do not depend on this choice up to homotopy. We leave this dependence implicit.

For example, if W is a rank-n vector bundle, its classifying map factors through a map $\xi_n\colon B \to B\mathrm{O}_n$, and $B^W \simeq \Sigma^\infty \mathrm{Th}(B, W)$.

Definition 1.1.12. Let $\xi\colon B \to B\mathrm{O}$ be a symmetry type. For $n \geq 1$, let $\xi_n\colon B_n \to B\mathrm{O}_n$ be the pullback of $\xi\colon B \to B\mathrm{O}$ along $B\mathrm{O}_n \to B\mathrm{O}$. Let $V_n \to B\mathrm{O}_n$ and $V \to B\mathrm{O}$ denote the tautological vector bundle, resp. the tautological stable vector bundle. By convention, $V \to B\mathrm{O}$ has rank zero.

(1) The Thom spectra $M\xi_n$, resp. $M\xi$, are the Thom spectra of $\xi_n^* V_n \to B_n$, resp. $\xi^* V \to B$.

(2) The *Madsen-Tillmann spectra* [**MT01, MW07**] $MT\xi_n$, resp. $MT\xi$, are the Thom spectra of $\xi_n^*(-V) \to B_n$, resp. $\xi^*(-V) \to B$.

If ξ is a G-structure as in Example 1.1.3, we will write MG, MTG, etc., rather than $M\xi$ and $MT\xi$.

Remark 1.1.13. Some Thom spectra go by many names. The notation $\mathbb{RP}^\infty_n$ denotes $(B\mathrm{O}_1)^{nV_1}$, and similarly $\mathbb{CP}^\infty_n \coloneqq (B\mathrm{SO}_2)^{nV_2}$. Thus, for example, $\Sigma^2 MTSO_2$, $\Sigma^2 MTU_1$, and $\Sigma^2\mathbb{CP}^\infty_{-1}$ all refer to $(B\mathrm{SO}_2)^{2-V_2}$.

Thom spectra were originally introduced to study bordism.

Theorem 1.1.14 (Pontrjagin [**Pon50, Pon55**], Thom [**Tho54**, Théorème IV.8])**.** *There is an isomorphism $\Omega_n^\xi \xrightarrow{\cong} \pi_n(MT\xi)$.*

Pontrjagin and Thom considered a few specific symmetry types; the idea to consider general symmetry types is due to Lashof [**Las63**]. The homotopy groups of $M\xi$ are the bordism groups of manifolds with a ξ-structure on their normal bundle. There is another symmetry type $\xi^\perp\colon B \to B\mathrm{O}$ such that $M\xi^\perp \simeq MT\xi$ and $M\xi \simeq MT\xi^\perp$, and $\xi^\perp$ is defined to be the composition

$$B \xrightarrow{\xi} B\mathrm{O} \xrightarrow{-1} B\mathrm{O}, \tag{1.1.15}$$

where -1 denotes the negation map from the H-group structure. For some symmetry types, $\xi \simeq \xi^\perp$, e.g. for O, SO, Spin, and Spin^c, but this operation exchanges Pin$^+$ and Pin$^-$.

Remark 1.1.16 (Ring structures)**.** Suppose that ξ has the *two-out-of-three property*, meaning that for two vector bundles $E, F \to X$, ξ-structures on any two of E, F, and $E \oplus F$ determine a ξ-structure on the third. Then direct product makes Ω_*^ξ into a graded ring. In this case, $MT\xi$ is an E_∞-ring spectrum,[6] and Theorem 1.1.14 upgrades to an isomorphism of graded rings. This applies in particular to MTO, $MTSO$, $MTSpin$, and $MTSpin^c$.

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[6]This was first proven by May-Quinn-Ray-Tornehave [**May77**, §IV.2], then later a different way by Ando-Blumberg-Gepner [**ABG10**, Example 6.22]. For a construction of the ring structure, see [**Sch**, Example 2.8].

Lemma 1.1.17. *$MTPin^+$, $MTPin^-$, and $MTPin^c$ cannot be ring spectra. More generally, if E is a spectrum with $\pi_0 E \cong \mathbb{Z}/2$ and $\pi_i(E)$ not a $\mathbb{Z}/2$-vector space, then E has no ring structure.*

The second sentence implies the first: for $\xi = \mathrm{Pin}^\pm$ or Pin^c, $\Omega_0^\xi \cong \mathbb{Z}/2$, but $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$, $\Omega_2^{\mathrm{Pin}^c} \cong \mathbb{Z}/4$, and $\Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}/16$.

Proof. If E is a ring spectrum, $\pi_*(E)$ is a graded ring with unit 1 in degree 0. Thus $\pi_0(E)$ is a ring and for all n, $\pi_n(E)$ is a $\pi_0(E)$-module. $\square$

Remark 1.1.18 (Module structures)**.** If ξ and ξ' are such that ξ-structures satisfy the two-out-of-three property and a ξ-structure on E and a ξ'-structure on F induce a ξ'-structure on $E \oplus F$, then $\Omega_*^{\xi'}$ is a module over Ω_*^ξ and $MT\xi'$ is an $MT\xi$-module spectrum, and Theorem 1.1.14 is compatible with this module structure. For example, $\mathrm{pin}^\pm$ bordism is a module over spin bordism, spin^c bordism is a module over spin bordism, and pin^c bordism is a module over spin^c bordism.

Remark 1.1.19 (Reinhardt bordism)**.** There is an analogue of Theorem 1.1.14 interpreting the homotopy groups of $MT\xi_n$ topologically: for example, in degree n, this is the *Madsen-Tillmann bordism group* $\Omega_n^{MT\xi_n}$, or the *Reinhardt bordism group*, or the *vector field bordism group*, or the *SKK bordism group*, or the *Lorentz bordism group*, which is defined to be the group completion of the commutative monoid of n-dimensional ξ-manifolds under disjoint union, modulo the bordism relation where M bounds if it bounds a compact $(n+1)$-dimensional ξ-manifold W and there is a nonvanishing vector field on W which is the outward normal on M. These bordism groups were first studied by Reinhardt [**Rei63**].

Unlike ordinary ξ-bordism, the group completion step is needed: without it, these would only be commutative monoids.

Bökstedt-Svane [**BS14**] show how to interpret the rest of the homotopy groups $\pi_k(MT\xi_n)$ as bordism groups.

Let $\xi\colon B \to B\mathrm{O}$ be a symmetry type and $\xi(X)$ be the symmetry type $B \times X \to B\mathrm{O}$, where the map is trivial on the space X. The corresponding Thom spectrum is $MT\xi \wedge X_+$, so as Atiyah discovered [**Ati61a**], the assignment from X to the $\xi(X)$-bordism groups is the generalized homology theory for the spectrum $MT\xi$. Concretely, $\Omega_*^\xi(X) \coloneqq \Omega_*^{\xi(X)}$ is the bordism theory of ξ-manifolds with a map to X, where saying that "Y bounds M" means not just that $\partial Y = M$, but that the map $M \to X$ extends to a map $Y \to X$.

Remark 1.1.20. That bordism is a generalized homology theory gives us some useful tools for computation, such as Mayer-Vietoris sequences[7] and the Atiyah-Hirzebruch spectral sequence. We will review the computational tools we need towards the end of this section.

1.1.3. The Thom isomorphism and orientations. In addition to capturing bordism-theoretic information in homotopy, Thom spaces and spectra also capture useful information in cohomology.

Proposition 1.1.21 (Thom isomorphism)**.** *Let $\pi\colon V \to X$ be an oriented rank-n vector bundle and A be an abelian group. Then there is a class $U \in \widetilde{H}^n(\mathrm{Th}(X, V); A)$ and an isomorphism*

$$\Phi\colon H^*(X;A) \xleftarrow[\pi^*]{\cong} H^*(V;A) \xrightarrow[(U\cdot)]{\cong} \widetilde{H}^{*+n}(\mathrm{Th}(X,V);A) \tag{1.1.22}$$

of $H^*(X;A)$-modules. If A is a $\mathbb{Z}/2$-vector space, V does not need to be oriented.

U is called the *Thom class*, and Φ the *Thom isomorphism*. Φ is natural with respect to change of coefficients and for pullbacks of vector bundles.

Corollary 1.1.23. *Proposition 1.1.21 gives rank-zero isomorphisms $H^*(X;A) \xrightarrow{\cong} \widetilde{H}^*(X^{\pm(V-\mathrm{rank}(V))};A)$ with the same hypotheses on V and A.*

This is because these Thom spectra are suspension spectra of Thom spaces. More generally, Thom spectra know twisted cohomology.

Definition 1.1.24. Let A be an abelian group, X be a connected space, and $\alpha \in H^1(X;\mathbb{Z}/2)$. Then A_α denotes the local system on X given by the $\mathbb{Z}[\pi_1(X)]$-module with underlying abelian group A and in which $g \in \pi_1(X)$ acts on A by $(-1)^{\alpha(g)}$, where we interpret α as a map $\pi_1(X) \to \mathbb{Z}/2$ under the identification $H^1(X;\mathbb{Z}/2) \cong \mathrm{Hom}(\pi_1(X), \mathbb{Z}/2)$.

Proposition 1.1.25 (Twisted Thom isomorphism)**.** *Let π be a rank-n vector bundle and A be an abelian group. Then there is a class $U \in \widetilde{H}^n(\mathrm{Th}(X,V);A)$ and an isomorphism of $H^*(X;A)$-modules*

$$\Phi\colon H^*(X;A_{w_1(V)}) \xrightarrow{\cong} \widetilde{H}^{*+n}(\mathrm{Th}(X,V);A) \tag{1.1.26}$$

defined analogously to (1.1.22). The analogue of Corollary 1.1.23 is also true.

[7] We do not really use the Mayer-Vietoris sequence in this thesis. For some examples where it is used to compute bordism groups for physics applications, see [**STY18, DH20, DDHM**].

Generalized orientation theory [**May77, ABG+14a, ABG+14b**] generalizes this story to generalized cohomology. For the rest of this section, fix a symmetry type ξ with the two-out-of-three property (e.g. O, SO, Spin), so that $MT\xi$ has the structure of an E_∞-ring spectrum.

Definition 1.1.27. Let R be a ring spectrum. A ξ*-orientation* of R is a homomorphism of ring spectra $MT\xi \to R$.

Theorem 1.1.28 (Generalized Thom isomorphism)**.** *Let R be a ring spectrum with a ξ-orientation and $V \to X$ be a vector bundle with ξ-structure. Then there is a Thom class $U \in \widetilde{R}^n(\mathrm{Th}(X,V))$ inducing a Thom isomorphism $\Phi\colon R^*(X) \overset{\cong}{\to} \widetilde{R}^{*+n}(\mathrm{Th}(X,V))$. The analogue of Corollary 1.1.23 is also true.*

In fact, for $X^{\pm(V-\mathrm{rank}(V))}$, the Thom isomorphism can be implemented as maps of spectra:

$$X^{\pm(V-\mathrm{rank}(V))} \wedge R \xrightarrow{\simeq} X_+ \wedge R \tag{1.1.29a}$$

$$\mathrm{Map}(X^{\pm(V-\mathrm{rank}(V))}, R) \xrightarrow{\simeq} \mathrm{Map}(X_+, R), \tag{1.1.29b}$$

which upon taking π_* recovers the usual R-(co)homology Thom isomorphisms [**MR81**].[8]

Therefore, one may define twisted generalized R-(co)homology with respect to a vector bundle V as the (untwisted) R-(co)homology of $X^{V-\mathrm{rank}(V)}$. This perspective, generalized from vector bundles to spherical fibrations, is taken up by Ando-Blumberg-Gepner [**ABG10**].

Example 1.1.30. The usual Thom isomorphism on ordinary cohomology is a special case of this construction. Let A be a commutative ring; then there are maps of ring spectra

$$MTSO \longrightarrow HA \tag{1.1.31a}$$

$$MTO \longrightarrow H\mathbb{Z}/2. \tag{1.1.31b}$$

The second map is identified with the map $\Omega_0^{\mathrm{O}} \to \mathbb{Z}/2$ which counts the number of points of a closed 0-manifold mod 2; the first map admits a similar description as a signed point count. The maps in (1.1.31) implement the Thom isomorphism as stated in Proposition 1.1.21 and Corollary 1.1.23.

Example 1.1.32 (Atiyah-Bott-Shapiro)**.** Atiyah-Bott-Shapiro [**ABS64**] produced maps

$$MTSpin \longrightarrow ko \tag{1.1.33a}$$

$$MTSpin^c \longrightarrow ku, \tag{1.1.33b}$$

[8] In this generality, this result is proven by Rudyak [**Rud98**, Theorem V.1.15].

where ko, resp. ku denotes connective real, resp. complex K-theory, and Joachim [**Joa04**] and Ando-Hopkins-Rezk [**AHR10**, Theorem 6.1] show these maps are E_∞-ring maps. That is, spin vector bundles are oriented for ko-theory, and spin^c bundles for ku-theory. Composing with the connective cover maps $ko \to KO$ and $ku \to KU$ gives spin, resp. spin^c orientations for KO-, resp. KU-theory. These maps may be understood as a homotopical version of taking the index of the Dirac operator on a family of spin or spin^c manifolds.

Example 1.1.34 (Tautological orientations)**.** For any symmetry type ξ satisfying the two-out-of-three property, the identity map defines a ξ-orientation of $MT\xi$. Surprisingly, this is occasionally useful for computations, e.g. in §4.4 and §4.5.

1.1.4. Important computational results. The theorems below are homotopy equivalences of spectra, but not of ring spectra.

Theorem 1.1.35 (Thom [**Tho54**, Théorème IV.12])**.** *There is an equivalence of spectra*

$$MTO \simeq H\big(\mathbb{Z}/2[x_i \mid i \neq 2^j - 1]\big), \tag{1.1.36}$$

where $|x_i| = i$.

So $H^*(X;\mathbb{Z}/2)$ determines $\Omega_*^{\mathrm{O}}(X)$. (1.1.36) induces an isomorphism of ring structures on π_*, though it is not an equivalence of ring spectra!

Theorem 1.1.37.

(1) (Thom [**Tho54**, Théorème IV.17, Corollaire IV.18]*)* $MTSO \wedge H\mathbb{Q} \simeq H(\mathbb{Q}[x_{4i} \mid i \geq 1])$, *where* $|x_{4i}| = 4i$. $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$ *is generated as a ring by the classes of complex projective spaces.*

(2) (Wall [**Wal60**]*)* $MTSO$ *is* 2*-locally additively equivalent to a wedge of copies of* $H\mathbb{Z}$ *and* $H\mathbb{Z}/2$ *beginning*

$$H\mathbb{Z} \vee \Sigma^4 H\mathbb{Z} \vee \Sigma^5 H\mathbb{Z}/2 \vee \Sigma^8 H\mathbb{Z} \vee \Sigma^8 H\mathbb{Z} \vee \cdots \tag{1.1.38}$$

(3) (Brown-Peterson [**BP66**, Theorem 1.3]*) If* p *is odd,* $MTSO$ *is* p*-locally additively equivalent to a wedge of suspensions of Brown-Peterson spectra.*

The free summands in Ω_*^{SO} are not all generated by complex projective spaces. The equivalence of spectra in (1) again induces an isomorphism of rings on π_*, though it is not an equivalence of ring spectra.

Theorem 1.1.39 (Anderson-Brown-Peterson [**ABP67**])**.** *MTSpin is* 2*-locally equivalent to a wedge of suspensions of ko,* $H\mathbb{Z}/2$*, and* $\tau_{\geq 2}ko$ *beginning*

$$ko \vee \Sigma^8 ko \vee \Sigma^8(\tau_{\geq 2}ko) \vee \cdots \tag{1.1.40}$$

The map MTSpin $\to$ *ko given by this splitting followed by projection onto the first factor can be identified with the Atiyah-Bott-Shapiro map* (1.1.33a).

Here $\tau_{\geq 2}$ denotes the 2-connective Postnikov truncation functor. If we work rationally or p-locally for odd p, the forgetful map $MTSpin \to MTSO$ is an isomorphism, so the only interesting case is at 2.

There are many variants of these bordism theories. In particular one often encounters bordism for symmetry types which are similar to spin structures. Often these can be reexpressed as the spin bordism of some space or spectrum X. This is one way to calculate $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$, a fact we make extensive use of in Chapter 3. We will need one general theorem.

Theorem 1.1.41 (Anderson-Brown-Peterson [**ABP67**])**.** $MTSpin^c$ *is* 2*-locally additively equivalent to a wedge of suspensions of ku and* $H\mathbb{Z}/2$ *beginning*

$$ku \vee \Sigma^4 ku \vee \Sigma^8 ku \vee \Sigma^8 ku \vee \Sigma^{10} H\mathbb{Z}/2 \vee \cdots \tag{1.1.42}$$

In degrees 59 *and below, the degrees of these summands can be read off of Bahri-Gilkey's table* [**BG87a**, p. 5].

The map $MTSpin^c \to ku$ *given by this splitting followed by projection onto the first factor can be identified with the Atiyah-Bott-Shapiro map* (1.1.33b).

At odd primes, there is an equivalence $MTSpin^c \to MTSO \wedge (B\mathrm{U}_1)_+$.

1.1.5. The Adams spectral sequence. Theorems 1.1.39 and 1.1.41 have the very convenient consequence that the Adams spectral sequence for computing the homotopy groups of $MTSpin \wedge X$ or $MTSpin^c \wedge X$ for a space or bounded-below spectrum X is much easier than in the general case. There is a general change-of-rings theorem, where if $\mathcal{B}$ is a graded Hopf algebra, $\mathcal{C} \subset \mathcal{B}$ is a graded Hopf subalgebra, and M and N are graded $\mathcal{B}$-modules, then there is a natural isomorphism

$$\mathrm{Ext}_{\mathcal{B}}^{s,t}(\mathcal{B} \otimes_{\mathcal{C}} M, N) \xrightarrow{\cong} \mathrm{Ext}_{\mathcal{C}}^{s,t}(M, N). \tag{1.1.43}$$

When $X = ko \wedge Y$ or $ku \wedge Y$, this greatly simplifies the E_2-page of the Adams spectral sequence. Inside the mod 2 Steenrod algebra $\mathcal{A}$, define the subalgebras $\mathcal{A}(1) \coloneqq \langle \mathrm{Sq}^1, \mathrm{Sq}^2 \rangle$ and $\mathcal{E}(1) \coloneqq \langle Q_0, Q_1 \rangle$;[9] then,

[9] These generators are given in two different bases of $\mathcal{A}$; the relations between them are $Q_0 = \mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$.

Stong [**Sto63**] showed $\widetilde{H}^*(ko;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{Z}/2$ and Adams [**Ada61**] showed $\widetilde{H}^*(ku;\mathbb{Z}/2) \cong \mathcal{A} \otimes_{\mathcal{E}(1)} \mathbb{Z}/2$. Both $\mathcal{A}(1)$ and $\mathcal{E}(1)$ are Hopf subalgebras of $\mathcal{A}$ so (1.1.43) says we need only consider

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(\widetilde{H}^*(X;\mathbb{Z}/2),\mathbb{Z}/2) \Longrightarrow \widetilde{ko}_{t-s}(X)^\wedge_2 \tag{1.1.44a}$$

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(\widetilde{H}^*(X;\mathbb{Z}/2),\mathbb{Z}/2) \Longrightarrow \widetilde{ku}_{t-s}(X)^\wedge_2. \tag{1.1.44b}$$

This line of reasoning, first used by Davis [**Dav74**], is by now a standard trick in algebraic topology.[10] For further reading, we recommend the paper of Beaudry-Campbell [**BC18**], who go into detail about how to define and calculate these Ext groups and work several examples over $\mathcal{A}(1)$. There are fewer worked examples of (1.1.44b) in the literature; see Bruner-Greenlees [**BG03**], Nguyen [**Ngu09**], Francis [**Fra11**, §5] and Al-Boshmki [**AB16**] for closely related calculations.

Our notation is standard in the $\mathcal{A}(1)$-case, but since examples for $\mathcal{E}(1)$ are sparser, we record here a few notational conventions for working with $\mathcal{E}(1)$-modules and this spectral sequence at the prime 2. When we draw $\mathcal{E}(1)$-modules, we will use solid straight lines to denote Q_0-actions and dashed curved lines to denote Q_1-actions. Therefore, for example, $\mathcal{E}(1)$ as a module over itself looks like Figure 2.

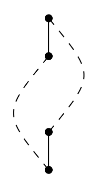

FIGURE 2. A diagram of the algebra $\mathcal{E}(1)$.

For any $\mathcal{E}(1)$-module M, $H^{*,*}(\mathcal{E}(1)) \coloneqq \mathrm{Ext}_{\mathcal{E}(1)}^{*,*}(\mathbb{Z}/2,\mathbb{Z}/2)$ acts on $\mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(M,\mathbb{Z}/2)$, analogously to the case of $\mathcal{A}(1)$-modules; if $M = \widetilde{H}^*(X;\mathbb{Z}/2)$, then just as over $\mathcal{A}(1)$, tracking this action through the Adams spectral sequence provides information about the action of ku_* on $\widetilde{ku}_*(X)$. Differentials are equivariant for this action, just like for the Adams spectral sequence over $\mathcal{A}(1)$. Since $\mathcal{E}(1)$ is an exterior algebra, Koszul duality provides an isomorphism of bigraded algebras

$$H^{*,*}(\mathcal{E}(1)) \cong \mathbb{Z}/2[h_0, v_1], \tag{1.1.45}$$

where $|h_0| = (1,1)$ and $|v_1| = (1,3)$ [**BC18**, Example 4.5.6]. We will denote an h_0-action by a vertical line, and a v_1-action by a lighter diagonal line. Like for ko, h_0 lifts to multiplication by 2; v_1 lifts to the action of the Bott element $\beta \in ku_2$ [**BG03**, §2.1].

[10]See Bruner-Greenlees [**BG10**, §1.13] for more on the history of this and other methods of computing ko-homology.

We will often write $\mathrm{Ext}_{\mathcal{A}(1)}(M)$ for $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(M,\mathbb{Z}/2)$, and similarly for $\mathcal{E}(1)$; when it is clear which subalgebra we are working over, we will just write $\mathrm{Ext}(M)$.

By now there is a large body of work using the Adams spectral sequence, especially over $\mathcal{A}(1)$, to compute generalized (co)homology groups for the purpose of studying invertible field theories or invertible phases. This includes [**Sto86**, **Kil88**, **Hil09**, **Fra11**, **FH16a**, **Cam17**, **BC18**, **GPW18**, **Guo18**, **FH19b**, **WW19a**, **WW19b**, **WWZ19**, **DL20a**, **DL20b**, **DL20c**, **GOP+20**, **KPMT20**, **LOT20**, **LT20**, **WW20a**, **WW20b**, **WW20c**, **WWZ20**].

1.1.6. The Atiyah-Hirzebruch spectral sequence.

The (homological) Atiyah-Hirzebruch spectral sequence [**AH61**] for ξ-bordism has signature

$$E^2_{p,q} = \widetilde{H}_p(X;\Omega^\xi_*) \Longrightarrow \Omega^\xi_{p+q}(X). \tag{1.1.46}$$

In general, using the Atiyah-Hirzebruch spectral sequence can feel different depending on application-specific details, so we point the reference-minded reader to García-Etxebarria-Montero [**GEM19**, §2.2.2, §3] for an introduction and some examples which may be helpful.

There are many references using the Atiyah-Hirzebruch spectral sequence to generalized homology or cohomology groups for the purposes of studying invertible field theories or invertible phases, such as [**Kil88**, **Edw91**, **Mon15**, **Cam17**, **KT17**, **Mon17**, **Hsi18**, **SdBKP18**, **SSG18**, **Ste18**, **STY18**, **SXG18**, **Xio18**, **ET19**, **FH19a**, **GEM19**, **MM19**, **OSS19**, **Shi19**, **TY19**, **BLT20**, **DGL20**, **DH20**, **DL20c**, **ETS20**, **GOP+20**, **HH20**, **HKT20**, **Hor20**, **HTY20**, **JF20a**, **JF20b**, **KPMT20**, **LOT20**, **LT20**, **SFQ20**, **Tho20**, **TW20**, **WW20b**, **Yu20**, **DGG21**, **JFY21**, **KLST21**].

We use a few other spectral sequences in our computations, but only for one-off computations, so we address them when we get to them.

1.1.7. Two more useful spectra.

In addition to the Thom spectra we defined above, we will repeatedly need a few more spectra.

Definition 1.1.47 (Brown-Comenetz [**BC76**])**.** Let A be an injective abelian group. An *(A-valued) Brown-Comenetz dual of the sphere spectrum* I_A is a spectrum representing the generalized cohomology theory

$$(I_A)^n(X) := \mathrm{Hom}(\pi_n(X), A). \tag{1.1.48}$$

Brown-Comenetz considered $A = \mathbb{Q}/\mathbb{Z}$; we mostly use $A = \mathbb{C}^\times$, for which this duality provides a version of character duality for spectra.[11] When A is a field of characteristic 0, the natural map $I_A \to HA$ is a weak equivalence.

One can choose I_A naturally in A.

Definition 1.1.49 (Anderson [**And69, Yos75**]). The *Anderson dual of the sphere*, denoted $I_\mathbb{Z}$, is the fiber of the map $I_\mathbb{C} \to I_{\mathbb{C}^\times}$ induced by the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^\times$.

$I_\mathbb{Z}$ satisfies the universal property that for any spectrum X, there is a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\pi_{n-1}(X), \mathbb{Z}) \longrightarrow [X, \Sigma^n I_\mathbb{Z}] \longrightarrow \mathrm{Hom}(\pi_n(X), \mathbb{Z}) \longrightarrow 0, \tag{1.1.50}$$

and this characterizes $I_\mathbb{Z}$ up to homotopy equivalence. (1.1.50) splits, but not naturally, implying a non-natural isomorphism from $[X, \Sigma^n I_\mathbb{Z}]$ to the direct sum of the torsion summand of $\pi_{n-1}(X)$ and the free summand of $\pi_n(X)$. We often use this fact implicitly, calculating $\pi_*(X)$ but depending on the reader to rearrange it into $[X, \Sigma^* I_\mathbb{Z}]$. For more on $I_\mathbb{Z}$ and its appearance in this context, see Freed-Hopkins [**FH16a**, §5.3, §5.4].

1.2. Topological field theories

We begin by defining topological field theories as mathematically formalized by Atiyah [**Ati88**], inspired by Segal's approach to conformal field theory [**Seg88**].

"Definition" 1.2.1. A *symmetric monoidal category* is a category C together with data of a bifunctor $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$, an element $1 \in \mathsf{C}$ called the *unit*, and additional data implementing associativtiy and commutativity of $\otimes$ and the fact that $1 \otimes - \cong - \otimes 1 \cong -$, subject to some coherence conditions.

For example, associativity is implemented via an *associator*, a natural isomorphism

$$\alpha\colon (- \otimes -) \otimes - \xrightarrow{\simeq} - \otimes (- \otimes -), \tag{1.2.2}$$

which is required to satisfy the *pentagon equation*, which guarantees that the different ways to rearrange the parentheses in a fourfold tensor product are coherent. We will not list all of these extra data and conditions; it is common to think of a symmetric monoidal category as "a category C with an associative, commutative tensor product $\otimes$, a unit 1, and some coherence data and conditions that we will not worry about." But for a complete definition, see Mac Lane [**ML71**, §XI.1].

[11] We do not really use this duality perspective: see Freed-Teleman [**FT18**, §9.1] and Liu [**Liu20**] for applications to electric-magnetic duality.

Example 1.2.3. Let k be a field.

(1) The usual tensor product on Vect_k is part of a symetric monoidal structure. The unit is k.

(2) If $\mathrm{char}(k) \neq 2$, the symmetric monoidal category of *super vector spaces*, denoted sVect_k, is the category of $\mathbb{Z}/2$-graded vector spaces with its usual tensor product, but with the commutativity data implementing $a \otimes b = (-1)^{|a||b|} b \otimes a$. The unit is k in even grading.

Example 1.2.4. Fix a symmetry type $\xi\colon B \to B\mathrm{O}$ and a dimension n. The *bordism category* $\mathsf{Bord}_{n,n-1}^{\xi}$ is the category C whose objects are closed $(n-1)$-manifolds with ξ-structure and whose morphisms $M \to N$ are diffeomorphism classes of data of a compact n-manifold X with ξ-structure and an identification $\partial X \xrightarrow{\cong} M \amalg (-N)$ as ξ-manifolds. The identity maps are cylinders.

$\mathsf{Bord}_{n,n-1}^{\xi}$ has a symmetric monoidal structure given by disjoint union, with the empty $(n-1)$-manifold as the unit.

"Definition" 1.2.5. Let C and D be symmetric monoidal categories. A *symmetric monoidal functor* $Z\colon \mathsf{C} \to \mathsf{D}$ is a functor sending $1_{\mathsf{C}} \mapsto 1_{\mathsf{D}}$ and such that $Z(x \otimes y) \cong Z(x) \otimes Z(y)$.

Again, this is not the whole definition, which asks for more, including a natural isomorphism between $Z(- \otimes -)$ and $Z(-) \otimes Z(-)$ and compatibility of this with the data implementing associativty, commutativity, etc. for C and D. You can read the full story in Mac Lane [**ML71**, §XI.2], though it is common to think of symmetric monoidal functors as "functors sending the unit to the unit and commuting with tensor product."

Definition 1.2.6 (Atiyah [**Ati88**], Segal [**Seg88**])**.** A *topological field theory* is a symmetric monoidal functor $Z\colon \mathsf{Bord}_{n,n-1}^{\xi} \to \mathsf{Vect}_k$.

The *dimension* of Z is n. Objects of $\mathsf{Bord}_{n,n-1}^{\xi}$ are $(n-1)$-dimensional; it is common to call $n-1$ the *space dimension* and n the *spacetime dimension*. If M is a closed $(n-1)$-manifold, the vector space $Z(M)$ is generally called the *state space* of M. A closed n-manifold X is canonically a bordism $\varnothing \to \varnothing$, so $Z(X)$ is a linear map $\mathbb{C} \to \mathbb{C}$, identified with a complex number; this number is called the *partition function* of Z on X, and we denote it by $Z(X)$.

Remark 1.2.7. There are a few different approaches to motivating Definition 1.2.6 from physics: for example, to a bordism X from M to N, a TFT Z attaches a linear map $Z(X)\colon Z(M) \to Z(N)$. One sometimes thinks of this as time-evolution of states on M to states on N, though if X is not a cylinder this is an imperfect analogy.

Alternatively, one can think of a TFT as encoding how partition functions behave on manifolds with boundary. In general quantum field theory, determining the partition function on M requires a choice of boundary data on ∂M, which is often a state in the Hilbert space of states on ∂M. The assignment "boundary data on ∂M to a partition function of M" defines a linear map $Z(M)\colon Z(\partial N) \to \mathbb{C} = Z(\varnothing)$, and this does make sense for nonidentity bordisms. Finally, gluing bordisms implements locality of the partition function, as we discuss further in §1.2.1.

1.2.1. Extended topological field theory. The Atiyah-Segal formalization of TFT is designed to encode the *locality* of quantum field theory, a principle that information in QFT, such as the partition function, can be computed on a manifold by chopping the manifold into pieces, computing something on those pieces, then recovering the partition function from those computations. Definition 1.2.6 only encodes part of this: decomposing a manifold into bordisms represents one direction of locality, but we ought to be able to decompose those bordisms further into manifolds with corners, and so on.

Extended topological field theory solves this problem by categorifying. Roughly speaking, a (weak) k-category is an algebraic structure like a category, but in which there are 2-morphisms between morphisms, 3-morphisms between 2-morphisms, and so on, up to level k. There are many different ways to make this precise; see Leinster [**Lei02**] for examples and references.

The idea to use higher categories to generalize the Atiyah-Segal definition of TFT was not introduced in a single paper, but was "in the air" in the mid-1990s, appearing in the work of many, including Baez-Dolan [**BD95**], Crane-Yetter [**CY99**], Freed [**Fre93, Fre94, Fre99**], Fukaya [**Fuk94**], Kapranov [**Kap95**], and Lawrence [**Law93**]. There is a symmetric monoidal k-category $\mathsf{Bord}_{n,n-k}^{\xi}$, called the *bordism k-category*, whose objects are closed $(n-k)$-manifolds with ξ-structure, whose morphisms are bordisms with ξ-structure between them, whose 2-morphisms are bordisms between those bordisms, again with ξ-structure, and so on up to degree k: the k-morphisms are diffeomorphism classes of n-dimensional bordisms between bordisms between … between bordisms, all equipped with ξ-structure.[12] We will exclusively take $k = 1$ (nonextended TFT, the original definition) in this thesis, except in Chapter 3, where we use $k = 2$ (sometimes called *once-extended TFT*); see that chapter for references relevant to the construction and study of bicategories and once-extended TFT. When $k = n$, we sometimes denote $\mathsf{Bord}_{n,0}^{\xi}$ by Bord_{n}^{ξ}.

Definition 1.2.8. A *k-extended TFT* is a symmetric monoidal functor $Z\colon \mathsf{Bord}_{n,n-k}^{\xi} \to \mathsf{C}$, where C is some symmetric monoidal k-category. When $k = n$, this is called a *fully extended TFT*.

[12] It is also common to work with bordism (∞, k)-categories, as constructed by Lurie [**Lur09b**] and Calaque-Scheimbauer [**CS19**], for which important definitions and constructions are generally easier.

Different researchers use different targets C, though generally we want $\Omega^{k-1}\mathsf{C} \simeq \mathsf{Vect}_{\mathbb{C}}$, so that by restricting a k-extended field theory to codimension 0 and 1, this recovers Definition 1.2.6. For example, when $k = 2$, one could take the Morita 2-category of associative, unital $\mathbb{C}$-algebras, or the 2-category of small $\mathbb{C}$-linear categories.

The idea that fully extended TFTs are fully local was originally an ansatz, but Grady-Pavlov [**GP20**, Theorem 1.0.1] prove a version of this as a theorem.

1.2.2. Truncated topological field theories. Often in the study of topological field theory, one is interested in something which behaves almost like an n-dimensional TFT, but which is not defined on most n-manifolds; instead, one only knows how to evaluate it on cylinders and mapping tori, leading to the notion of a truncated TFT. For example, when we study the low-energy behavior of lattice Hamiltonian systems in Chapters 2 and 3, we will find truncated TFTs rather than usual TFTs.

Definition 1.2.9. The *m-truncated* (k-extended, n-dimensional) *bordism category* $\tau_{<m}\mathsf{Bord}^{\xi}_{n,n-k}$ is the symmetric monoidal sub-k-category of $\mathsf{Bord}^{\xi}_{n,n-k}$ containing all objects and ℓ-morphisms for $\ell < m$, but only the invertible ℓ-morphisms for $\ell \geq m$. An *m-truncated TFT* is a symmetric monoidal functor $\tau_{<m}\mathsf{Bord}^{\xi}_{n,n-k} \to \mathsf{C}$ for some symmetric monoidal k-category C.

For the most part, $m = k$, i.e. we only throw out noninvertible morphisms in codimension 0. When we refer to a truncated TFT in this thesis, we mean k-truncated unless otherwise specified. For such a truncated TFT Z, we have only very few partition functions, those of mapping tori of diffeomorphisms together with automorphisms of the ξ-structure. In fact, all information in the TFT is contained in positive codimension: if $f\colon M \to M$ is an automorphism of a ξ-manifold M and M_f denotes the associated mapping torus, then the partition function $Z(M_f)$ is the trace of the map $Z(M) \to Z(M)$ defined by the mapping cylinder of f.

If $Z\colon \mathsf{Bord}^{\xi}_{n,n-k} \to \mathsf{C}$ is any TFT, restricting to $\tau_{<m}\mathsf{Bord}^{\xi}_{n,n-k}$ defines a truncated TFT, which we call the *truncation* of Z and denote $\tau_{<m}Z$. If m is clear from context we will just write τZ.

We use truncated TFTs to study lattice Hamiltonian systems. In these systems, relativistic symmetry is broken: we have chosen an arrow of time so that we can write down a Hamiltonian. Therefore we cannot expect to place the system on an arbitrary spacetime manifold, so any TFT extracted from such a system (typically through the low-energy behavior) is going to be something like a relative TFT. This application of relative TFTs is suggested by Kong-Wen [**KW14**] and Freed-Hopkins [**FH16a**, §7.3].

Remark 1.2.10. Truncated TFTs appear in several other places in the TFT literature. Freed-Teleman [FT14] formalize anomalous QFTs as boundary systems to truncated invertible field theories, and Fuchs-Schaumann-Schweigert [FSS19] and Müller-Szabo [MS18] study this idea in detail. Skein TFTs, used for studying quantum invariants of 3-manifolds, are constructed as truncated TFTs by Walker [Wal] and Gunningham-Jordan-Safronov [GJS19]. Ben-Zvi and Nadler describe the Betti geometric Langlands program [BZN18, BZN21] as an equivalence of two truncated 4d TFTs, and truncated TFTs appear in the work of Ben-Zvi, Nadler, and their collaborators [BZN09, BZFN10, BZBJ18a, BZBJ18b, BZGN19]. Other examples appear in papers of Douglas, Schommer-Pries, Snyder [DSPS13], Brochier-Jordan-Snyder [BJS18, §1.5], Kirillov Jr. and Tham [KT20], and Walker [Wal21].

1.3. Invertible field theories and bordism

Invertible TFTs are special examples of TFTs which are almost, but not quite, trivial. This makes them good examples to study directly, as in Chapters 3 and 4, or to use to build more general TFTs using the finite path integral, as in §2.2.

Definition 1.3.1. Let $(\mathsf{C}, \otimes, \mathbf{1})$ be a symmetric monoidal k-category and x be an object in C. Then x is *invertible* if there is $x^{-1} \in \mathsf{C}$ such that $x \otimes x^{-1} \cong \mathbf{1}$.

The space of data x^{-1} and equivalences $x \otimes x^{-1} \simeq \mathbf{1}$ is contractible.

Definition 1.3.2 (Freed-Moore [FM06, Definition 5.7])**.** An *invertible TFT* is an invertible object in the symmetric monoidal k-category of TFTs.

Example 1.3.3 (Euler theories)**.** Let $\lambda \in \mathbb{C}^\times$. The *Euler theory* $Z_\lambda\colon \mathsf{Bord}_{n,n-1}^{\mathrm{O}} \to \mathsf{Vect}_{\mathbb{C}}^{\times}$ is an invertible TQFT which to every object assigns $\mathbb{C}$, and to every morphism $X\colon M_1 \to M_2$ assigns multiplication by $\lambda^{\chi(X,M_1)}$. These compose properly because the Euler characteristic satisfies a gluing formula.

Invertible TFTs can be characterized in a few equivalent ways.

- Invertibility of a TFT Z implies that for any closed $(n-1)$-dimensional ξ-manifold M, $Z(M)$ is $\otimes$-invertible (so in $\mathsf{Vect}_{\mathbb{C}}$, a one-dimensional vector space) and for any bordism X, $Z(X)$ is invertible under composition. In particular, all partition functions are nonzero.
- Schommer-Pries [SP18] shows that in many cases, it suffices to check invertibility on T^n with its various ξ-structures.

Next, we speedrun the classification of invertible field theories. This classification is due to Freed-Hopkins-Teleman [FHT10], who reduce it to a question in stable homotopy theory.

Definition 1.3.4. A *Picard k-groupoid* is a symmetric monoidal k-category which is a k-groupoid (all morphisms in all degrees are invertible under composition) and such that every object is $\otimes$-invertible.

Let C be a (small) symmetric monoidal k-category. We can extract two Picard k-groupoids from C.

- The *Picard k-groupoid of units* $\mathsf{C}^\times$ is the subcategory of $\otimes$-invertible objects, composition-invertible 1-morphisms between those objects, composition-invertible 2-morphisms between those 1-morphisms, and so forth.
- The *Picard k-groupoid completion* $\overline{\mathsf{C}}$ is formed from C by formally adding inverses for all objects and morphisms. This has the universal property that if D is a Picard k-groupoid, any map $\mathsf{C} \to \mathsf{D}$ factors uniquely through $\overline{\mathsf{C}}$.

In particular, a TFT $Z\colon \mathsf{Bord}_{n,n-k}^\xi \to \mathsf{C}$ is invertible iff it factors through $\mathsf{C}^\times \hookrightarrow \mathsf{C}$. The universal property mentioned above implies that the space of invertible TFTs (i.e. the subspace of the space of TFTs $\mathsf{Bord}_{n,n-k}^\xi \to \mathsf{C}$) is naturally homotopy-equivalent to the space of symmetric monoidal functors

$$\overline{\mathsf{Bord}_{n,n-k}^\xi} \longrightarrow \mathsf{C}^\times. \tag{1.3.5}$$

If D is a Picard k-groupoid, then the geometric realization of its nerve is an E_∞-space (under tensor product) which is grouplike (all objects in D are $\otimes$-invertible). Therefore it defines a connective spectrum, which we call the *classifying spectrum* of D and denote $|\mathsf{D}|$. This is a complete invariant of D, up to equivalence of Picard k-groupoids.

Theorem 1.3.6 (Stable homotopy hypothesis (Moser-Ozornova-Paoli-Sarazola-Verdugo [MOP+20]))**.** *There is an equivalence of ∞-categories between the ∞-category of Picard k-groupoids and the ∞-category of spectra whose homotopy groups vanish outside of* $[0, k]$.[13]

Remark 1.3.7. For small k, this was proven earlier. When $k = 1$, the stable homotopy hypothesis was a folklore theorem for a while: proofs or sketches can be found in [**BCC93, HS05, Dri06, Pat12, JO12, GK14**]. For $k = 2$, the stable homotopy hypothesis was proven by Gurski-Johnson-Osorno [**GJO19**].

This implies that the abelian group of isomorphism classes of invertible TFTs of ξ-manifolds is naturally isomorphic to

$$\pi_0\mathrm{Map}(|\overline{\mathsf{Bord}_{n,n-k}^\xi}|, |\mathsf{C}^\times|). \tag{1.3.8}$$

[13]The proof in [**MOP+20**] uses the *Tamasmani model* [**Tam99**] for k-categories. Haugseng [**Hau15**, Example 6.20] has shown that this is equivalent to the standard models of weak k-categories. I thank Alexander Campbell for bringing this reference to my attention.

Therefore we are interested in identifying the classifying spectra of $\overline{\mathsf{Bord}^{\xi}_{n,n-k}}$ and $\mathsf{C}^\times$ for the choices of C we care about. For the Picard groupoid completions of bordism categories, this is work of Galatius-Madsen-Tillmann-Weiss [**GMTW09**] and Nguyen [**Ngu17**] in the 1-categorical case and Schommer-Pries [**SP17**] in the fully general (∞, n)-categorical case. Recall the definition of Madsen-Tillmann spectra from Definition 1.1.12.

Theorem 1.3.9 (Galatius-Madsen-Tillmann-Weiss [**GMTW09**], Nguyen [**Ngu17**], Schommer-Pries [**SP17**])**.** *There is a natural homotopy equivalence*

$$|\overline{\mathsf{Bord}^{\xi}_{n,n-k}}| \simeq \tau_{0:k}\Sigma^k MT\xi_n. \tag{1.3.10}$$

For the codomain, the classification depends on one's choice of C.

Lemma 1.3.11. *Let K be a field; then $|\mathsf{Vect}_K^\times| \simeq \Sigma HK^\times$.*

Proof. Invertible K-vector spaces are all isomorphic to K, and the invertible linear maps $K \to K$ are identified with $K^\times$. $\square$

We will see in §3.3.3.3 that $|\mathsf{sVect}_{\mathbb{C}}^\times|$ is a truncation of $I_{\mathbb{C}^\times}$, a spectrum we disucssed in §1.1.7.

For once-extended TFT, we also understand $|\mathsf{C}^\times|$ fairly well. Bartlett, Douglas, Schommer-Pries, and Vicary [**BDSPV15**, Appendix A] show that for many choices of deloopings C of Vect_K, $|\mathsf{C}^\times| \simeq \Sigma^2 HK^\times$; we will show in §3.3.3.3 that $|\mathsf{sAlg}_{\mathbb{C}}^\times| \simeq \tau_{\geq 0}\Sigma^2 I_{\mathbb{C}}^\times$. Much less is known in higher category number.

Remark 1.3.12. There are other related approaches to the classification of invertible TFTs by Kreck-Stolz-Teichner (unpublished) and Rovi-Schoembauer [**RS18**].

1.3.0.1. *Reflection-positive invertible TFTs.* For applications in physics, we demand more from our invertible TFTs. Two of the pillars of quantum field theory are locality and unitarity. Extended TFT handles locality, but it is not yet clear what unitarity should mean for general TFTs. See Johnson-Freyd [**JF17**] for one approach in the once-extended case, and conjecturally in full generality.

In quantum field theory, *reflection positivity* is the analogue of unitarity after Wick-rotating to Euclidean signature. Freed-Hopkins [**FH16a**] define and classify reflection positive invertible TFTs using Borel-equivariant stable homotopy theory; their definition of reflection positive invertible TFTs is a model for the invertible TFTs that describe unitary quantum systems.

Recall from §1.1.7 that $I_{\mathbb{Z}}$ denotes the Anderson dual of the sphere.

Theorem 1.3.13 (Freed-Hopkins [**FH16a**, Theorem 1.1])**.** *The abelian group of reflection positive, n-dimensional, invertible field TFTs on manifolds with ξ-structure is isomorphic to the torsion subgroup of $[MT\xi, \Sigma^{n+1} I_{\mathbb{Z}}]$.*

This has a concrete consequence in field-theoretic terms. The torsion subgroup of $[E, \Sigma^{n+1} I_{\mathbb{Z}}]$ is naturally identified with the torsion subgroup of $\mathrm{Hom}(\pi_n(E), \mathbb{C}^\times)$; for $MT\xi$ this is the group of torsion $\mathbb{C}^\times$-valued bordism invariants of n-dimensional ξ-manifolds. Theorem 1.3.13 extends to the statement that if Z is a reflection positive invertible TFT, its partition function is a $\mathbb{C}^\times$-valued bordism invariant, and the isomorphism in Theorem 1.3.13 sens Z to its partition function.

Therefore to compute groups of isomorphism classes of reflection positive invertible TFTs, one has to compute bordism groups. Most appearances of bordism in this thesis are to this end.

Remark 1.3.14 (Non-topological invertible field theories)**.** Freed-Hopkins [**FH16a**, Conjecture 8.37] go further and conjecture that the entirety of $[MT\xi, \Sigma^{n+1} I_{\mathbb{Z}}]$ classifies all invertible field theories, topological or not. Assuming this conjecture, the partition functions of the non-topological invertible field theories are the secondary invariants associated to bordism invariants $\Omega_{n+1}^{\xi} \to \mathbb{Z}$, such as η-invariants or Chern-Simons invariants. See Freed [**Fre19**, Lecture 9] for more information.

In almost all cases of interest in this thesis, $[MT\xi, \Sigma^{n+1} I_{\mathbb{Z}}]$ is torsion, and we can ignore this detail. However, it will come up in the classification of rotation-equivariant invertible phases in §4.4.3.

1.4. Hamiltonian systems modeling topological phases

This thesis applies methods from topological field theory to mathematical questions motivated by condensed-matter physics. In this section, we discuss these physics applications. This section is more heuristic than the others: mathematically formalizing some of the physics notions covered in this section is an open problem. The definitions in this section are not all mathematical.

1.4.1. Topological phases of matter. Physicists studying condensed-matter systems discovered that some of them have unusual properties; for example, these systems can have "quasiparticles," localized excitations that behave like particles but are not actually particles, instead consequences of the highly entangled nature of the electrons in the system. These quasiparticles can have unusual statistics: for example, some of these systems are lower-dimensional and can therefore have quasiparticles which are neither bosons nor fermions. These systems are examples of *topological phases of matter* — a topological phase is something like an equivalence class of systems with the same physical properties, though a precise definition is not yet available.[14]

Condensed-matter theorists are interested in producing models for these systems and classifying them. This question has been the subject of a great deal of research in the last 15 years. There is not yet a standard model for topological phases, and producing a general theoretical definition is a significant open problem. However, by studying examples, physicists have learned some properties that the eventual definition must satisfy. We will first go over some of these ideas, and then introduce the lattice Hamiltonian approach to studying topological phases.

When studying a topological phase of matter, it is important to specify a collection of symmetries that should not be broken. For example, some topological phases have a time-reversal symmetry, and others do not. The classifications of topological phases with time-reversal symmetry and without it are expected to be different — there can be ways to deform one system into another that break time-reversal symmetry, so these phases are inequivalent as phases with time-reversal symmetry but equivalent in its absence. Specifying the collection of symmetries is akin to specifying the symmetry type of a TFT.

There is expected to be a tensor product operation on topological phases, called *stacking*, in which one considers two phases separately on the same medium, with no interactions between them. The two phases should have the same symmetries.

Heuristic Definition 1.4.1. A *symmetry-protected topological (SPT)* phase, or a *short-range entangled (SRE)* phase, or an *invertible phase*, is a topological phase of matter which is invertible under stacking, i.e. there is some other topological phase with the same symmetries and such that when those two phases are stacked, the resulting system is in the trivial phase.

Under stacking, SPTs for a given dimension and collection of symmetries, SPTs form an abelian group. Computing these abelian groups has been the focus of significant research in condensed-matter theory in the past decade.

Remark 1.4.2. The first definitions of SPTs required them to be nontrivial in the presence of the symmetries in question, but trivializable in the absence of these symmetries. This kills the abelian group structure, and also could miss some phases which are nontrivial even after forgetting the symmetry.[15]

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[14] It is not even clear how to precisely define or model physical systems — for example, what kinds of Hamiltonians should we allow when considering topological phases?

[15] The ansatz that SPTs are classified by invertible field theories implies that there is such a phase in dimension $4+1$, whose low-energy field theory is the reflection positive invertible TFT with partition invariant w_2w_3. This phase is investigated by Fidkowski-Haah-Hastings-Tantivasadakarn [**FHHT20, FHH20**], and a related crystalline SPT phase is studied by Huang [**Hua20**].

Heuristic Definition 1.4.3. A topological phase of matter which has a collection of symmetries but is not necessarily invertible is called a *symmetry-enriched topological (SET)* phase.

Heuristic Definition 1.4.4. A topological phase of matter in which the collection of symmetries acts nontrivially on space is called a *crystalline topological phase* or *crystalline SET phase*. If it is invertible, it is called a *crystalline SPT phase*.

We will relate these notions to topological field theory below in §1.4.3.

Remark 1.4.5. Physicists also study *free fermion phases*, including *topological insulators* and *topological superconductors*. These also have interesting mathematical connections to topology, but are out of scope of this thesis. See Freed-Hopkins [**FH16a**, §9.2] for more information. In the context of free fermion phases, the phases we consider are generally called *interacting phases*.

1.4.2. A Hamiltonian formalism for topological phases of matter. Physicists study topological phases of matter from several different viewpoints. We will focus on the lattice Hamiltonian formalism: studying a phase by discretizing space and writing down a state space and Hamiltonian using that discrete data. This is a common approach to modeling topological phases. Before we give a detailed example in Example 1.4.7, we summarize a few of the general features.

A lattice Hamiltonian model for a topological phase is roughly a procedure for assigning to a manifold with a triangulation the data of a complex Hilbert space $\mathcal{H}$, called the *state space*, and a self-adjoint operator $H\colon \mathcal{H} \to \mathcal{H}$ called the *Hamiltonian*. Sometimes the combinatorial structure is more general or less general than a triangulation; sometimes the manifold has additional structure encoded combinatorially, such as a spin structure or principal bundle, and the state space and Hamiltonian may depend on this data.

Both the state space and Hamiltonian should be local and determined solely from the data of the triangulation. Though precise definitions of these desiderata are not yet available in full generality, "local" should mean that the state space is a tensor product of finite-dimensional "local state spaces" on each simplex, and that the Hamiltonian is a sum of operators each of which vanishes on the local state spaces for simplices not contained within some compact set, and that the radii of these compact sets, in the graph distance on simplices, should be uniformly bounded above. We do not attempt to give a precise definition in this thesis.

We also ask for the Hamiltonian to be *gapped*, meaning that the difference between the smallest two eigenvalues of the Hamiltonian is uniformly bounded below. Again, characterizing this condition precisely is still an open question. Once these notions are formalized, there is conjectured to be a space of gapped lattice systems (for a fixed dimension and symmetry type), and topological phases are precisely the connected components of this space. Said differently, changing the Hamiltonian by smooth deformations that do not close the spectral gap should not change the physics of the system or invariants of the phase, such as the low-energy TFT that we discuss below.

In the Hamiltonian formalism, the symmetries of the phase are encoded in this discrete data: for example, crystalline phases with a C_4 rotation symmetry can be modeled on a square lattice with a Hamiltonian that is invariant under $\pi/2$ rotations. Understanding how to express these symmetries using the Hamiltonian is understood in many cases but not in complete generality.[16]

Two Hamiltonian lattice models which can be defined using the same triangulation can be stacked: say their state spaces are $\mathcal{H}_1$ and $\mathcal{H}_2$ and their Hamiltonians are $H_1\colon \mathcal{H}_1 \to \mathcal{H}_1$ and $H_2\colon \mathcal{H}_2 \to \mathcal{H}_2$. Stacking these two systems means considering the system with state space $\mathcal{H}_1 \otimes \mathcal{H}_2$ and Hamiltonian

$$H_1 \otimes 1 + 1 \otimes H_2 \colon \mathcal{H}_1 \otimes \mathcal{H}_2 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{1.4.6}$$

Example 1.4.7 (Toric code)**.** The toric code is the model organism of this field, originally defined by Kitaev [**Kit03**] and since studied from many perspectives in the mathematics and physics literature and generalized in many directions: defining it on nonorientable surfaces [**FM01**]; generalizing it to manifolds of any dimension [**FML02**]; placing the spins on k-cells, rather than edges [**DKLP02**]; considering a fermionic variant [**GWW14**]; changing whether it is even a gauge theory at all [**BMCA13**]; and adding global symmetries [**BBJ+16, HBFL16, LV16**]. In this thesis, we will not consider most of these generalizations.

Let n be the spacetime dimension. For X a CW complex, let X^k denote its k-skeleton and $\Delta^k(X)$ denote its set of k-cells. We let $\mathrm{Bun}_{\mathbb{Z}/2}(X)$ denote the groupoid of principal $\mathbb{Z}/2$-bundles on X and, for a subspace $Y \subset X$, let $\mathrm{Bun}_{\mathbb{Z}/2}(X, Y)$ denote the groupoid of principal $\mathbb{Z}/2$-bundles $P \to X$ together with a trivialization $\xi\colon P|_Y \xrightarrow{\cong} \underline{\mathbb{Z}/2}$ on Y.

Definition 1.4.8. The data $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(X^1, X^0)$ determines a function $\mathrm{spin}_{(P,\xi)}\colon \Delta^1(X) \to \mathbb{Z}/2$: if e is a 1-cell of X, $P|_e$ descends to a principal bundle $P' \to e/\partial e$, where we use the trivialization of P on ∂e to identify the fibers. Then $\mathrm{spin}_{(P,\xi)}(e)$ is 0 if P' is trivial, and 1 if it is nontrivial. In other words, if $\partial e = \{v, w\}$, we can compare $\xi(v)$ and $\xi(w)$ by parallel-transporting along e; then $\mathrm{spin}_{(P,\xi)}(e)$ is their difference. The function $\mathrm{spin}_{(P,\xi)}$ determines (P, ξ) up to isomorphism.

The groupoid of fields for the toric code is $\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$, and the state space assigned to M is $\mathcal{H} \coloneqq \mathbb{C}[\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)]$, the vector space of complex-valued functions on the groupoid of fields.[17] Given

[16]For example, Gaiotto-Kapustin [**GK16**] study Hamiltonian models for fermionic phases and argued that they require a discretization of a spin structure as constructed by Cimasoni-Reshetikhin [**CR07**] or Budney [**Bud13**].

[17]The space of functions on a groupoid G is defined to be the vector space of functions $\pi_0 G \to \mathbb{C}$.

$(P,\xi) \in \pi_0 \operatorname{Bun}_{\mathbb{Z}/2}(M^1, M^0)$, let $\delta_{(P,\xi)} \in \mathcal{H}$ be the function sending $(P,\xi) \mapsto 1$ and all nonisomorphic (P',ξ') to 0. The set

$$\{\delta_{(P,\xi)} \mid (P,\xi) \in \pi_0 \operatorname{Bun}_{\mathbb{Z}/2}(M^1, M^0)\} \tag{1.4.9}$$

is a basis for $\mathcal{H}$; endow $\mathcal{H}$ with the inner product for which it is an orthonormal basis.

Given a 0-cell v of M, let $A_v \colon \mathcal{H} \to \mathcal{H}$ denote the *shift operator* at v: if $\psi \in \mathcal{H}$ and $(P,\xi) \in \operatorname{Bun}_{\mathbb{Z}/2}(M^1, M^0)$, let $\xi + \delta_v$ denote the section of P on M^0 which is identical to ξ except on v, where its value is $\xi(v) + 1$. Then,

$$A_v(\psi)(P,\xi) \coloneqq \psi(P, \xi + \delta_v). \tag{1.4.10a}$$

Given a 2-cell f of M, let $B_f \colon \mathcal{H} \to \mathcal{H}$ be multiplication by the holonomy around ∂f:

$$B_f(\psi)(P,\xi) \coloneqq (-1)^{\mathrm{Hol}_P(f)} \psi. \tag{1.4.10b}$$

There are operators associated to each 2-cell f and each 0-cell v, called *face operators*, resp. *vertex operators*:

$$H_f \coloneqq \frac{1 - B_f}{2} \tag{1.4.11a}$$

$$H_v \coloneqq \frac{1 - A_v}{2}, \tag{1.4.11b}$$

and the Hamiltonian assigned to M is

$$H_{\mathrm{TC}} \coloneqq \sum_{v \in \Delta^0(M)} H_v + \sum_{f \in \Delta^2(M)} H_f. \tag{1.4.12}$$

Remark 1.4.13. The original definition of the toric code looked different, replacing (P,ξ) with the function $\mathrm{spin}_{(P,\xi)} \colon \Delta^1(M) \to \mathbb{Z}/2$ it defines. The state space is the free complex vector space on the finite set of these functions. The analogues of A_v and B_f for $v \in \Delta^0(M)$ and $f \in \Delta^2(M)$ are

$$A'_v \coloneqq \prod_{e : v \in \partial e} \sigma^x_e \tag{1.4.14a}$$

$$B'_f \coloneqq \prod_{e \in \partial f} \sigma^z_e. \tag{1.4.14b}$$

Here, σ^x and σ^z are the Pauli operators

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4.15}$$

The state space $\mathcal{H}$ can be identified with the tensor product of local state spaces $\mathcal{H}_e \coloneqq \mathbb{C} \cdot \{0, 1\}$ over each 1-cell e, and the notation σ_e^x and σ_e^z means these operators act on $\mathcal{H}_e$ by the matrices in (1.4.15), and by the identity on the remaining tensor factors.

We can identify A'_v with A_v by observing that switching the trivialization for (P, ξ) over v amounts to switching the value of $\mathrm{spin}_{(P,\xi)}$ on any 1-cell e adjacent to v, which is the action by σ_e^x. To identify B_f and B'_f, observe that the holonomy of (P, ξ) around ∂f is the product of the spins on the 1-cells in ∂f.

Proposition 1.4.16 (Kitaev [**Kit03**], Freedman-Meyer-Luo [**FML02**])**.**

(1) The Hamiltonian H_{TC} is self-adjoint.

(2) The H_f and H_v operators are projectors, and pairwise commute.

(3) $\mathrm{Spec}(H_{\mathrm{TC}}) \subset \mathbb{Z}_{\ge 0}$, and 0 is always an eigenvalue.

Proof sketch. Using the identifications of A_v with A'_v and B_f with B'_f, A_v and B_f are products of real symmetric matrices, hence are themselves real symmetric matrices; therefore H_v and H_f are too. Therefore H is a sum of real symmetric matrices, proving part (1).

Part (2) is directly analogous to Kitaev's original proof in dimension $n - 1 = 2$ [**Kit03**]; see [**FML02**] for the generalization to higher dimensions.

Part (3) follows because the eigenvalues of A_f and B_v are in $\{\pm 1\}$, so the eigenvalues of H_f and H_v are in $\{0, 1\}$. The function dual to the trivial bundle with the identity trivialization is an eigenvector for 0. $\square$

We will discuss the toric code further in Chapter 2, where we compare it to the GDS model and study their behavior at low energy.

Remark 1.4.17. The best-behaved Hamiltonians are sums of commuting projection operators on $\mathcal{H}$; these are (unsurprisingly) called *commuting-projector Hamiltonians*. Finding a commuting-projector Hamiltonian for a given phase can be significantly more difficult than finding an arbitrary Hamiltonian; for example, $(3 + 1)$d fermionic phases with a time-reversal symmetry squaring to fermion parity are believed to have a $\mathbb{Z}/16$ classification [**Kit13b, FCV13, MFCV14, WS14, YX14, Kit15, KTTW15, FH16a, TY16, Wit16, SSR17a, WG20**], but finding a commuting-projector Hamiltonian for any generator of that $\mathbb{Z}/16$ is open.

If it is possible to model all topological phases with lattice Hamiltonians, that suggests an approach to the classification of topological phases: fixing a dimension and collection of symmetries, presumably there is a space of lattice Hamiltonians, and the gapped lattice Hamiltonians form a subspace. The space of topological phases is then π_0 of that subspace. Making this precise is, as usual, an open problem. We will take an alternative approach to the classification question.

1.4.3. The low-energy approach. Our approach to classifying and studying topological phases is through their behavior at low energy: *it is believed that the low-energy limit of a gapped topological phase of matter is a topological field theory,*[18] *and that this is a complete invariant.* It is also expected that all TFTs arise in this way (at least, for some target categories) [**FH16a**, **Gai17**, **RW18**, **FT18**]. As with most applications of TFT to physics, these TFTs should be fully extended and unitary, though hidden in this "should" is a large amount of open questions, including formulating unitarity for general noninvertible TFTs.

Under this hypothesized equivalence, consider a lattice Hamiltonian model and its associated low-energy TFT Z. If M is a manifold with a triangulation (or whichever combinatorial structure we need to define the state space and Hamiltonian on M), the Hamiltonian H is a self-adjoint operator on a finite-dimensional vector space, so has a smallest eigenvalue λ.[19] The space of *ground states* of the lattice model is the eigenspace for λ.

Part of the content of the statement that "Z is the low-energy TFT for this lattice model" is that the space of ground states on M is isomorphic to the state space $Z(M)$. This is a strong statement — it implies that the dimension of the space of ground states cannot depend on the triangulation! See §2.3.2 for this calculation for the toric code.

The low-energy ansatz includes a correspondence between the collections of symmetries as expressed in physics and symmetry types of TFTs: for example, fermionic phases with a time-reversal symmetry squaring to 1 are believed to correspond to pin^- topological field theories, in the sense that the low-energy limit of such a phase is expected to be a pin^- TFT.

Under this ansatz, stacking corresponds to the tensor product of TFTs. Therefore the low-energy limit of an SPT (ignoring crystalline symmetries for now) should be a reflection positive invertible field theory, and the corresponding classifications should match.

Remark 1.4.18. We would like to go farther and determine a way to extract the entire TFT Z from the lattice model. It is expected that Z is a fully extended TFT, as it is a low-energy description of a quantum system, so it should be fully local. In principle, and in a few examples, one can use information such as extended operators in the lattice model to read off the value of Z on lower-dimensional manifolds, as discussed and implemented in [**BK12**, **BD19**, **BG20**, **BD21**]. But going upwards to codimension zero is harder. In

[18] In general, we need to allow topological theories tensored with invertible (but not necessarily topological) field theories. See [**FH16a**, §5.4]. This important subtlety does not arise in this thesis.

[19] Often H is normalized so that $\lambda = 0$. This is true for the toric code, by Proposition 1.4.16, as well as for the other lattice Hamiltonian models we consider in Chapters 2 and 3.

this thesis, we mostly do not address this question, focusing on the correspondence between state spaces and spaces of ground states, but in Chapter 2 we show how to extract the data of partition functions for mapping tori in some lattice models, extracted from a $\mathrm{Diff}(M)$-representation that we construct on the space of ground states of the lattice model on M.

Example 1.4.19. In the absence of additional symmetries, $(2+1)$-dimensional gapped topological phases of matter are believed to correspond to suitable equivalence classes of *spherical fusion categories*. Levin-Wen [**LW05**] wrote down a Hamiltonian lattice model given a spherical fusion category C. This data also defines a 3d oriented TFT called the *Turaev-Viro-Barratt-Westbury (TVBW)* model for C [**TV92, BW96**]. Kirrillov Jr [**Kir11**] showed that the ground states of the Levin-Wen Hamiltonian on closed surfaces are isomorphic to the state spaces of the TVBW model,[20] and Goosen [**Goo18**] generalized this to surfaces with boundary.

Example 1.4.20. Fidkowski-Kitaev [**FK10, FK11**] and Turner-Pollman-Berg [**TPB11**] found a $\mathbb{Z}/8$ classification of $(1+1)$d fermionic SPTs with a time-reversal symmetry squaring to 1. Under the ansatz, these correspond to reflection positive pin$^-$ invertible field theories; Freed-Hopkins [**FH16a**, (9.86)] relate this to $\mathrm{Hom}(\Omega_2^{\mathrm{Pin}^-}, \mathbb{C}^\times) \cong \mathbb{Z}/8$; this bordism group is calculated by Anderson-Brown-Peterson [**ABP69**] and Kirby-Taylor [**KT90b**]. We discuss this example in detail in Chapter 3.

We summarize the low-energy approach with a dictionary of predicted correspondences.

- A gapped topological phase of matter should be described at low energy by a TFT[21] with the same dimension and symmetry type.
- An SPT phase should correspond to an invertible TFT.
- The space of ground states of a lattice Hamiltonian on M, in any triangulation, should be isomorphic to the state space of the low-energy TFT on M.

We consider crystalline phases in Chapter 4, but this dictionary does not quite apply to them: the notion of a group G acting on space cannot be encoded in our definition of symmetry type. Kitaev [**Kit13a, Kit15**] posits that the groups of phases on G-spaces Y should fit together into a Borel-equivariant generalized homology theory. Freed-Hopkins [**FH19a**] use this to make an ansatz for some classes of crystalline SPT phases in terms of invertible field theories, and our Ansatz 4.1.19 builds on theirs.

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[20] Douglas, Schommer-Pries, and Snyder [**DSPS13**] use the cobordism hypothesis to study 3d oriented TFTs valued in the Morita 3-category of rigid $\mathbb{C}$-linear monoidal categories, and conjecture that all such TFTs are isomorphic to TVBW models.
[21] Possibly tensored with an invertible field theory.

1.5. Summary of results

1.5.1. Chapter 2. In this chapter, we study the generalized double semion (GDS) model of Freedman-Hastings [**FH16b**]. Freedman-Hastings studied the low-energy TFT of the GDS model, but did not determine it for odd n. We extract a truncated TFT from the spaces of ground states of this model and show it is isomorphic to the truncation of a TFT Z_{GDS} which we define as a slight generalization of Dijkgraaf-Witten theory. Conjecturally, this means that Z_{GDS} is the low-energy TFT of the GDS model.

We begin in §2.1 by defining the GDS model and studying a few of its properties. Freedman-Hastings wrote down this model as a spin liquid, and we reformulate it as a lattice gauge theory for the group $\mathbb{Z}/2$.

Then, in §2.2, we define a class of TFTs called $\mathbb{Z}/2$-*gauge-gravity theories* which generalized $\mathbb{Z}/2$-Dijkgraaf-Witten theories by allowing Stiefel-Whitney classes in the Lagrangian action. Choose $\beta \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$; then β defines a mod 2 characteristic class $\beta(M,P)$ on a manifold M with a principal $\mathbb{Z}/2$ bundle $P \to M$ by mapping to $B\mathrm{O}_n$ using TM and to $B\mathbb{Z}/2$ using P. Given β, we build a $\mathbb{Z}/2$-gauge-gravity theory Z_β in two steps. First, the Freed-Hopkins-Teleman classification of invertible TFTs [**FHT10**] implies there is an n-dimensional invertible TFT

$$Z_\beta^{\mathrm{cl}} \colon \mathsf{Bord}_{n,n-1}^{\mathrm{O}}(B\mathbb{Z}/2) \longrightarrow \mathsf{Vect}_{\mathbb{C}}^{\times}, \tag{1.5.1}$$

unique up to isomorphism, whose partition function is $\langle \beta(M,P), [M]\rangle$.[22] Second, we perform the finite path integral, summing over principal $\mathbb{Z}/2$-bundles to obtain a (generally noninvertible) TFT

$$Z_\beta \colon \mathsf{Bord}_{n,n-1}^{\mathrm{O}} \longrightarrow \mathsf{Vect}_{\mathbb{C}}. \tag{1.5.2}$$

In §2.3, we investigate the low-energy limits of the toric code and the GDS models. We develop a method to extract a truncated TFT from a lattice model satisfying some conditions; the state spaces of this truncated TFT are the spaces of ground states of the lattice model. Both the toric code and the GDS model meet these restrictions, and we compute the truncated TFTs this method associates to them. The following theorem is a combination of Theorems 2.3.5 and 2.3.19.

Theorem. *Let DW_0 denote $\mathbb{Z}/2$ finite gauge theory (sometimes also known as untwisted Dijkgraaf-Witten theory), and let Z_{GDS} denote the $\mathbb{Z}/2$-gauge-gravity theory for β equal to the degree-n piece of $w\alpha/(1+\alpha) \in H^*(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$, where α is the generator of $H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$.*

[22]Freed-Hopkins' classification of reflection-positive invertible TFTs [**FH16a**] implies this invertible TFT admits a reflection positive structure. It can also be extended. We do not need either of these properties in this thesis.

(1) Let L^{TC} denote the truncated TFT we extract from the toric code in §2.3. Then there is an equivalence of truncated TFTs $\tau\mathrm{DW}_0 \simeq L^{\mathrm{TC}}$.

(2) Let L^{GDS} denote the truncated TFT we extract from the GDS model in §2.3. Then there is an equivalence of truncated TFTs $\tau Z_{\mathrm{GDS}} \simeq L^{\mathrm{GDS}}$.

For the toric code, results along these lines were previously known, but for the GDS model when $n > 3$ is odd, this is new.

1.5.2. Chapter 3. The bordism group $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$, generated by $\mathbb{RP}^2$ with either of its pin$^-$ structures and detected by a complete invariant called the *Arf-Brown(-Kervaire)* invariant [**Bro71, KT90b**]

$$AB\colon \Omega_2^{\mathrm{Pin}^-} \longrightarrow \mu_8 \subset \mathbb{C}^\times, \tag{1.5.3}$$

where μ_8 denotes the abelian group of 8^{th} roots of unity. Therefore, by the Freed-Hopkins [**FH16a**] classification of reflection-positive invertible TFTs, there is a reflection-positive invertible TFT $Z_{AB}\colon \mathsf{Bord}_{2,0}^{\mathrm{Pin}^-} \to \mathsf{C}$, called the *Arf-Brown theory*, whose partition function is the Arf-Brown invariant. Here C is some symmetric monoidal bicategory with $\mathsf{C}^\times \simeq \tau_{\geq 0}\Sigma^2 I_{\mathbb{C}^\times}$. Z_{AB} is unique up to isomorphism. The purpose of this chapter, based on joint work with Sam Gunningham, is to study Z_{AB}, making explicit some aspects of Freed-Hopkins' classification theorem and its relation to SPT phases in condensed-matter physics.

The Arf-Brown invariant is a generalization of the more familiar Arf invariant of a spin surface. The Arf invariant admits three quite different-looking descriptions: one using a quadratic refinement of the intersection pairing; one using a mod 2 index of the spin Dirac operator; and one using KO-theory. Likewise, we present three equivalent definitions of the Arf-Brown invariant: in §3.2.1, we give Brown's definition of the Arf-Brown invariant using a $\mathbb{Z}/4$-quadratic enhancement of the intersection form [**Bro71**]. In §3.2.2, we give Zhang's interpretation of the Arf-Brown invariant as an η-invariant [**Zha94, Zha17**]. In §3.2.3, we give a new interpretation of the Arf-Brown invariant as a pushforward in twisted KO-theory, related to a similar construction of Distler-Freed-Moore [**DFM10**].

We then discuss the relationship between invertible TFTs and stable homotopy theory, going into more detail than we did in §1.3. Given a torsion bordism invariant $\alpha\colon \Omega_n^H \to \mathbb{C}^\times$, we show how to compute the data that the corresponding invertible TFT Z_α associates to closed manifolds in codimension 1 and 2 in terms of Postnikov invariants of the classifying spectrum for $\mathsf{C}^\times$, using work of Gurski-Johnson-Osorno-Stephan [**GJOS17**].

In order to study Z_{AB}, we need to identify a symmetric monoidal bicategory C such that $\mathsf{C}^\times \simeq \tau_{\geq 0}\Sigma^2 I_{\mathbb{C}^\times}$.

Proposition 3.3.22. *Let* $\mathsf{sAlg}_{\mathbb{C}}$ *denote the symmetric monoidal bicategory of complex superalgebras, with* 1*-morphisms as super-bimodules and* 2*-morphisms as bimodule homomorphisms. Then* $|\mathsf{sAlg}_{\mathbb{C}}^{\times}| \simeq \tau_{\geq 0}\Sigma^2 I_{\mathbb{C}^\times}$.

A closely related theorem appears in Freed's Vienna notes [**Fre12**, Theorem 1.52]. Therefore the Arf-Brown theory can be constructed with target $\mathsf{sAlg}_{\mathbb{C}}$; we do so in §3.4, in particular calculating its values on 0d and 1d pin$^-$ manifolds.

In §3.5, we apply the Arf-Brown theory to physics, studying the Majorana chain with its time-reversal symmetry. The classification of 2d fermionic SPTs with a time-reversal symmetry squaring to 1 is believed to be $\mathbb{Z}/8$, given by sending such a phase to its low-energy invertible field theory, and the relevant classification of invertible field theories is believed to be the group of 2d pin$^-$ reflection-positive TFTs, isomorphic to $\mathbb{Z}/8$. The Majorana chain is a lattice Hamiltonian model predicted to be a representative for one of the four generators of this $\mathbb{Z}/8$ of phases, and therefore the Arf-Brown theory, or some odd tensor multiple of it, should describe the Majorana chain at low energy. We investigate this by defining the Majorana chain on a pin$^-$ 1-manifold with a triangulation, encoding the pin$^-$ structure in additional discrete data. We then compute the space of ground states, and prove that these agree with the state spaces of Z_{AB}.

Corollary 3.5.30. *Assuming the ansatz from §1.4.3 that the low-energy TFT of a topological phase is a complete invariant, the low-energy TFT* Z *of the Majorana chain is a generator of the* $\mathbb{Z}/8$ *of deformation classes of reflection positive pin*$^-$ *invertible field theories. In particular, its deformation class is an odd multiple of the class of the Arf-Brown theory.*

1.5.3. Chapter 4. In this chapter, we provide a model for the classification of invertible phases on a G-space, prove that the groups of such phases are isomorphic to certain groups of invertible field theories; and make computations in several examples.

Kitaev [**Kit13a, Kit15**] proposed that the classification of invertible phases on a space Y should form a generalized homology theory, and Freed-Hopkins [**FH19a**] use this to make an ansatz computing groups of phases on G-spaces as Borel-equivariant generalized homology groups. We extend Freed-Hopkins' ansatz to the case where the G-symmetry on space can mix with the internal symmetry of the phase, so as to account for, for example, fermionic phases with a C_4 rotation symmetry, such that a full 2π rotation acts on fermions by -1. Such mixed symmetries have been studied in the physics literature where G is a group of rotations, reflections, inversions, or rotations and reflections.

Given a G-equivariant local system f of symmetry types over a G-space Y, we define *phase homology* groups $Ph_*^G(Y; f)$ in §4.1.2 using equivariant generalized Borel-Moore homology. The G-equivariant local system of symmetry types provides a model for the background data needed to define a class of phases whose symmetries mix with the G-action on spacetime, such as in the example above; we discuss this in more detail in §4.1.3. We then predict that phase homology groups model groups of equivariant phases.

ANSATZ 4.1.19. The group of G-equivariant invertible phases on Y for this data is isomorphic to the equivariant phase homology group $Ph_0^G(Y; f)$.

We next address the fermionic crystalline equivalence principle. Crystalline equivalence principles express the classification of *crystalline phases*, which are certain topological phases of matter in which a group of symmetries acts on space, in terms of classifications of phases without such a spatial symmetry. Thorngren-Else [**TE18**] were the first to study crystalline equivalence principles, with followup work by [**JR17, CW18, ET19, FH19a, ZWY$^+$20, ZYQG20**]; thanks to these authors, the equivalence principle for bosonic SPT phases (corresponding to symmetry types such as O or SO) is well-understood, and for fermionic crystalline SPTs (corresponding to Spin, Pin$^\pm$, etc.) it is understood in special cases.

We provide a general fermionic crystalline equivalence principle (FCEP) in Altland-Zirnbauer classes D and A, corresponding to symmetry types $H = \mathrm{Spin}$ and $H = \mathrm{Spin}^c$. The theorem identifies phase homology groups with groups of reflection positive invertible field theories, but with a twist: the symmetry types on the two sides of this equivalence do not match, and are exchanged. That is, given a representation $\lambda\colon G \to \mathrm{O}_d$ and some additional data, in Definitions 4.2.3 and 4.2.4 we define symmetry types modeling the cases of spinless and spin-1/2 fermions in a fermionic SPT with an internal G-symmetry (i.e. G does not act on space), and in Definition 4.2.2 we define G-equivariant local systems of symmetry types on $\mathbb{R}^d$ modeling the case of spinless and spin-1/2 fermions in a fermionic SPT, where G acts on space by λ.

THEOREM 4.2.8 (Fermionic crystalline equivalence principle). *Fixing data of G, H, λ, etc. as above, let $f_0, f_{1/2}$ denote the local systems of symmetry types for the case of spinless, resp. spin-1/2 fermions. Then $Ph_0^G(\mathbb{R}^d; f_0)$ is isomorphic to the group of deformation classes of d-dimensional IFTs for the spin-1/2 internal symmetry type, and $Ph_0^G(\mathbb{R}^d, f_{1/2})$ is isomorphic to the group of deformation classes of d-dimensional IFTs for the spinless internal symmetry type.*

Ansatz 4.1.19 interprets this as equivalences between classifications of crystalline SPT phases and ordinary SPT phases, where the way in which the G-symmetry mixes with fermion parity switches.

In §4.4 and §4.5, we use Theorem 4.2.8 and the Adams spectral sequence to compute phase homology groups for point group actions on $\mathbb{R}^n$. We consider reflections, inversions, rotations, D_{2n} acting both by rotations and reflections in 2d and rotations in 3d, pyritohedral symmetry, and both chiral and full tetrahedral, octahedral, and icosahedral symmetry. These phase homology groups correspond to groups of crystalline

SPTs, and some of the corresponding groups of crystalline SPTs have been studied in the physics literature. We compare our results with preexisting ones and find agreement. Some of our computations, particularly for three-dimensional point groups, correspond to phases not yet studied in the literature, so our computations are predictions.

In §4.6, we discuss phase homology groups for $\mathbb{Z}$ acting on $\mathbb{R}^d$ by a glide symmetry. Ansatz 4.1.19 relates these groups to groups of *glide SPTs*. Lu-Shi-Lu [**LSL17**] study glide SPTs and conjecture a formula for their classification; in Theorem 4.6.4 we prove the corresponding formula for phase homology groups.

1.5.4. Chapter 5. Say that two closed, connected, smooth 4-manifolds M and N are *stably diffeomorphic* if $M\,\#(S^2\times S^2)^{\#m}$ is diffeomorphic to $N\,\#(S^2\times S^2)^{\#n}$ for some $m,n\geq 0$. If M and N are closed, connected topological 4-manifolds, we define *stably homeomorphic* analogously. Kreck's modified surgery theory [**Kre99**] reduces the classification of 4-manifolds up to stable diffeomorphism or stable homeomorphism to a series of bordism questions, and one can work one fundamental group at a time, as we discuss further in §5.1. Our first theorem simplifies this bordism question for certain choices of $\pi_1(M)$ in the case that M is unorientable.

Theorem 5.2.1. *Suppose G is a finite group fitting into an extension*

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} P \longrightarrow 1, \tag{1.5.4}$$

where $|K|$ is odd and P is a 2-group, and suppose P acts trivially on $H^*(BK)$. For any unorientable virtual vector bundle $V\to BP$, φ induces an equivalence of Thom spectra $(BG)^{\varphi^*V}\xrightarrow{\simeq}(BP)^V$.

We implement this simplification in the simplest case: if G is the fundamental group of an unorientable manifold, then the description of loops as orientation-preserving or orientation-reversing defines a surjection $p\colon G\twoheadrightarrow\mathbb{Z}/2$. Therefore G cannot be finite of odd order, and the simplest case is $|G|\equiv 2 \bmod 4$. In this case, assuming $\mathbb{Z}/2$ acts trivially on $H^*(B\ker(p))$, we completely determine the stable diffeomorphism classification of unorientable 4-manifolds M with $\pi_1(M)\cong G$.

Theorem. *Let G be a finite group of order $2 \bmod 4$, and suppose that $\mathbb{Z}/2$ acts trivially on $H^*(B\ker(p))$.*

(1) There are fourteen equivalence classes of closed, connected, unorientable 4-manifolds M up to stable diffeomorphism: nine for which M is pin$^+$, one for which M is pin$^-$, and four for which M is neither.

(2) There are twenty equivalence classes of closed, connected, unorientable topological 4-manifolds M up to stable homeomorphism: ten for which M is pin$^+$, two for which M is pin$^-$, and eight for which M is neither.

This is a combination of Theorems 5.3.2, 5.3.5, 5.4.2 and 5.4.5. We also determine complete stable diffeomorphism and homeomorphism invariants in these cases. The classification for M neither pin^+ or pin^- can be extracted from work of Davis [**Dav05**, Theorem 2.3], but the other parts are new.

CHAPTER 2

The low-energy effective TFT of the generalized double semion model

The content of this chapter was published as [**Deb20**]. It has been lightly edited to be streamlined with the rest of the thesis.

2.0. Introduction

The classification of topological phases of matter is an active area of research in the theory of condensed-matter physics and in nearby mathematical fields. There are many different approaches to this classification problem (for an incomplete sample, see [**PTBO10**, **LG12**, **CGLW13**, **Kit13a**]), but from a mathematical point of view, a classification via low-energy limits is appealing: based on physical insights, it is believed that the low-energy effective theory of a gapped phase of matter is a topological quantum field theory (TFT), possibly tensored with an invertible theory, and that passage to the low-energy effective theory should send physically distinct phases to distinct TFTs [**FH16a**, **Gai17**, **RW18**, **FT18**]. As TFTs have a purely mathematical description due to Atiyah-Segal [**Ati88**, **Seg88**], this reframes the classification question within mathematics — though a systematic mathematical understanding of this physical ansatz relating lattice systems to effective field theories remains out of reach. Even at a physical level of rigor, it is not clear what the general definition of the low-energy effective theory of a lattice model should be, and without this it is impossible to rigorously verify the efficacy of the low-energy approach to classification in general. Nonetheless, there are many examples of lattice models in the physical and mathematical literature, and it is instructive to study what can be said about their low-energy effective theories in order to gain insight into the general picture. Some examples include [**Kir11**, **BK12**, **Cha14**, **ALW17**, **BCK$^+$17**, **CILT17**, **KMM21**].

In this paper, we investigate the low-energy effective theory of the generalized double semion (GDS) lattice model of Freedman-Hastings [**FH16b**], which exists in every dimension. Freedman and Hastings define the GDS model and study its spaces of ground states on different manifolds, showing that in even (spacetime) dimensions n they are isomorphic to the state spaces of the $\mathbb{Z}/2$-Dijkgraaf-Witten theory with Lagrangian equal to 0, but that for odd $n > 3$, they are not isomorphic to the state spaces of any $\mathbb{Z}/2$-Dijkgraaf-Witten theory. For every dimension n, we extract a truncated TFT L from the GDS model anddefine an n-dimensional

TFT $Z_{\mathrm{GDS}}\colon \mathsf{Bord}_{n,n-1}^{\mathrm{O}} \to \mathsf{Vect}_{\mathbb{C}}$; then we show that as truncated TFTs, $L \simeq \tau Z_{\mathrm{GDS}}$.[1] Along the way, we reformulate the GDS model as a lattice gauge theory with gauge group $\mathbb{Z}/2$: it is a theory formulated on manifolds with a triangulation, which plays the role that a Riemannian metric does in Wick-rotated quantum field theory. We find that, as for the toric code lattice model, the low-energy limit does not depend on the triangulation, and is described by the state spaces of a TFT. For both the toric code and GDS models, this TFT is a $\mathbb{Z}/2$-gauge theory, but unlike for the toric code, the GDS theory involves gravity, in that Stiefel-Whitney classes of the underlying manifold enter the effective action. This explains the above result of Freedman-Hastings that this TFT cannot be any $\mathbb{Z}/2$-Dijkgraaf-Witten theory when n is odd and greater than 3 [**FH16b**, Theorem 8.1].

The GDS model is closely analogous to the toric code; thus, throughout this paper, we will introduce ideas first for the toric code, which is simpler, and then turn to the GDS model. In §2.1, we define the GDS model (§2.1.1) in arbitrary dimension. Like the toric code from Example 1.4.7, the GDS model is a lattice model. Lattice models are discretized analogues of quantum field theories studied in condensed-matter physics: one puts a combinatorial structure, such as a CW structure or a triangulation, on a manifold, and formulates all data of the theory, including the fields and the Hamiltonian, in terms of this combinatorial structure. The toric code and GDS models are typically written as spin liquids, meaning the fields are functions from the edges of a lattice to $\{\uparrow, \downarrow\}$. We reformulate them as lattice gauge theories, describing equivalent models whose fields are discretizations of principal $\mathbb{Z}/2$-bundles.

In §2.2, we construct a class of TFTs called $\mathbb{Z}/2$-gauge-gravity theories. They generalize Dijkgraaf-Witten theories with gauge group $\mathbb{Z}/2$, but the Lagrangian includes Stiefel-Whitney classes of the underlying manifold in addition to characteristic classes of the principal $\mathbb{Z}/2$-bundle. First, in §2.2.1, we define "classical gauge-gravity theories," invertible TFTs of manifolds with a principal $\mathbb{Z}/2$-bundle. Then, in §2.2.2, we quantize these theories, summing over the groupoid of principal $\mathbb{Z}/2$-bundles to produce TFTs $Z_\beta\colon \mathsf{Bord}_{n,n-1}^{\mathrm{O}} \to \mathsf{Vect}_{\mathbb{C}}$ of unoriented manifolds given a cohomology class $\beta \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$.

In §2.3, we use these gauge-gravity TFTs to study the low-energy behavior of the GDS model. The Hamiltonian in the GDS model has spectrum contained within $\mathbb{Z}_{\geq 0}$, and the space of ground states of the GDS model on an $(n-1)$-manifold M is defined to be the kernel of the Hamiltonian for M. In examples arising in physics from topological phases of matter, the space of ground states often depends only on M, and not on the triangulation. When this occurs, it is expected that this extends to a TFT $Z\colon \mathsf{Bord}_{n,n-1}^{\mathrm{O}} \to \mathsf{Vect}_{\mathbb{C}}$, in that for any closed $(n-1)$-manifold M, $Z(M)$ is isomorphic to the space of ground states on M. First, in

[1]Recall the definition of truncated TFTs from §1.2.2. That L is a truncated TFT means that for every closed $(n-1)$-manifold M, we have a state space $L(M)$ with a $\mathrm{Diff}(M)$-action, which is the space of ground states of the GDS model on M. That $L \simeq \tau Z_{\mathrm{GDS}}$ means that there is a $\mathrm{Diff}(M)$-equivariant isomorphism $L(M) \cong Z_{\mathrm{GDS}}(M)$ for all M.

§2.3.1, we gather the data of the spaces of ground states of the GDS model into a *truncated TFT* L. For any closed $(n-1)$-manifold M, L assigns to M the space of ground states of the GDS model on M , and L also contains data of a $\mathrm{Diff}(M)$-action on $L(M)$, built using the data of the lattice model. We provide a method for some lattice models of constructing a $\mathrm{Diff}(M)$-action on the space of ground states of M and therefore producing a truncated TFT from the spaces of ground states of the lattice model. When we say we want to determine the low-energy TFT of a lattice model, we mean finding some truncated TFT Z that we understand and showing $Z \simeq L$. In §2.3.2, we implement this idea for the toric code, where we reprove the following known result.

Theorem 2.3.5. *Let L^{TC} be the truncated TFT the above method extracts from the toric code. If $\mathrm{DW}_0 \colon \mathsf{Bord}^{\mathrm{O}}_{n,n-1} \to \mathsf{Vect}_{\mathbb{C}}$ denotes the $\mathbb{Z}/2$-Dijkgraaf-Witten theory with Lagrangian equal to 0, then there is an equivalence of truncated TFTs $\tau\mathrm{DW}_0 \simeq L^{\mathrm{TC}}$.*

This implies that for every closed $(n-1)$-manifold M, the space of ground states of the toric code on M is isomorphic to $\mathrm{DW}_0(M)$ as $\mathrm{Diff}(M)$-representations. In §2.3.3, we turn to the GDS model, where we prove the main theorem. Let $\alpha \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$ denote the generator and $w \in H^*(B\mathrm{O}_n;\mathbb{Z}/2)$ denote the total Stiefel-Whitney class. In the graded ring $H^*(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2) \cong H^*(B\mathrm{O}_n;\mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H^*(B\mathbb{Z}/2;\mathbb{Z}/2)$, α is nilpotent, so $1+\alpha$ is invertible. Therefore we can form $w\alpha/(1+\alpha) \in H^*(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$, which is a sum of homogeneous elements of different degrees.

Theorem 2.3.19. *Let L^{GDS} be the truncated TFT the above method extracts from the GDS model. Let $\beta \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$ be the degree-n summand of $w\alpha/(1+\alpha)$. Then $\tau Z_\beta \cong L^{\mathrm{GDS}}$.*

Again, this means an isomorphism of state spaces equipped with $\mathrm{Diff}(M)$-representations. Because of this theorem, Z_β will also be denoted Z_{GDS}.

In §2.4, we provide some calculations with this low-energy TFT, allowing us to prove a comparison theorem with $\mathbb{Z}/2$-Dijkgraaf-Witten theories.

Theorem.

(1) In dimension 3, there is an isomorphism between Z_{GDS} and the $\mathbb{Z}/2$-Dijkgraaf-Witten theory with Lagrangian equal to the nonzero element of $H^3(B\mathbb{Z}/2;\mathbb{Z}/2)$.

(2) In any even dimension, there is an isomorphism between Z_{GDS} and DW_0.

(3) For odd $n \geq 5$, Z_{GDS} is distinct from all $\mathbb{Z}/2$-Dijkgraaf-Witten theories.

This theorem is a combination of Theorems 2.4.29, 2.4.31 and 2.4.32. Part (3) was first proven by [**FH16b**], as was (2) for state spaces.

2.1. The toric code and GDS models

Definition 2.1.1. Let X be a topological space with a CW structure Ξ. We let $\Delta^k(X)$ denote its set of k-cells and X^k denote its k-skeleton. When we need to make explicit that these are with respect to Ξ, we will write $\Delta^k(X;\Xi)$, resp. X^k_Ξ. If Π is a triangulation of X, we will also write $\Delta^k(X;\Pi)$ and X^k_Π for the k-simplices, resp. k-skeleton, of X with respect to Π.

When we need Ξ to be explicit, we will write $C^\Xi_k(X;A)$ (resp. $C^k_\Xi(X;A)$) for the group of cellular k-chains (resp. k-cochains) with coefficients in an abelian group A for the CW structure Ξ. We will employ analogous notation for cycles and cocycles, and for simplicial (co)chains and (co)cycles with respect to a given triangulation Π.

Throughout this chapter, we use the toric code as an extended example to introduce ideas that we then implement in the GDS model. Therefore it may be helpful to look back at Example 1.4.7 and compare it with our definition of the GDS model below.

2.1.1. Generalized double semion model.

Our main focus is the generalized double semion (GDS) model.

The double semion model for $n = 3$ was first studied by Freedman-Nayak-Shtengel-Walker-Wang [**FNS**$^+$**04**] and Levin-Wen [**LW05**, §VI.A], then generalized to all dimensions n by Freedman and Hastings [**FH16b**].[2] The name comes from the description of this theory in the case $n = 3$ as the lattice model associated to the Drinfeld double of the semion modular tensor category.[3]

Definition 2.1.2. Let M be a simplicial complex and c be a simplex of M.

- The *open star* of c, denoted $\mathrm{St}(c)$, is the subset of M consisting of all simplices whose closures contain c.
- The *closed star* of c, denoted $\overline{\mathrm{St}}(c)$, is the smallest subcomplex containing $\mathrm{St}(c)$.
- The *link* of c, denoted $S(c)$, is $\overline{\mathrm{St}}(c) \setminus \mathrm{St}(c)$.

For the GDS model, we need a neighborhood of v in between the open and closed stars of v.

[2] There are a few other generalizations of the double semion model in low dimensions [**vKBS13, LV16, OMD16, DOVMD18**], but we focus on Freedman-Hastings' construction.

[3] The semion modular tensor category is named such because the excitations in its corresponding lattice model are semions, anyonic quasiparticles with statistics intermediate between those of bosons and fermions. For $n > 3$, however, the name "generalized double semions" is somewhat of a misnomer, however: anyons cannot exist in dimension $n > 3$, because the braids that define their mutual statistics can be unlinked. See [**RW18**, §2.1]. It is also not clear that the theory is the double of another [**FH16b**, §1]. At least it is generalized.

Definition 2.1.3. Let M be a simplicial complex and e be a simplex of M. Define the 0-*clopen star* $\overline{\mathrm{St}}(0)(e)$ to be $\mathrm{St}(e) \cup \overline{\mathrm{St}}(e)^0$. That is, we include the 0-simplices of the closed star of e as well as all cells in the open star.

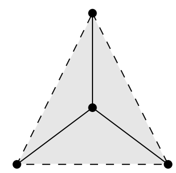


FIGURE 3. The 0-clopen star of a vertex in a simplicial structure on a surface.

As before, fix a dimension n; we proceed to define the state space and Hamiltonian that the GDS model assigns. In order to avoid pathologies, one cannot define the GDS model for an arbitrary CW structure.

Definition 2.1.4. A *triangulation* of a smooth manifold M is a simplicial complex K together with a homeomorphism $f\colon |K| \to M$; if for every simplex e of K, the restriction of f to $|e|$ is smooth, we say (K, f) is a *smooth triangulation*.

When defining the GDS model, we choose a smooth triangulation Π such that the 0-clopen star of every vertex is contractible.[4] We discuss in Remark 2.1.25 why restricting to such triangulations, rather than more general combinatorial structures such as CW structures, is necessary.

The GDS model assigns to every closed $(n-1)$-manifold M with such a triangulation a state space and Hamiltonian, like the toric code does; the state space is $\mathbb{C}[\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)]$ as for the toric code, and we proceed to define the Hamiltonian, which is similar to that of the toric code, but with an extra sign.

Definition 2.1.5. Let M be a closed $(n-1)$-manifold with a smooth triangulation such that the 0-clopen star of every vertex is contractible. Then, given $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$ and a 0-simplex v, there is a unique maximal extension of ξ to a subset of $\overline{\mathrm{St}}(0)(v)$; we denote that subset $Y_v'(P, \xi)$.

Definition 2.1.6. Let $v \in \Delta^0(M; \Pi)$ and $S(v)$ denote the link of v in the barycentric subdivision Π_1 of Π. Though $S(v)$ comes equipped with a triangulation $\Pi_1|_{S(v)}$, we define a new triangulation $\Pi_{S(v)}$ on $S(v)$. For $k \geq 0$, if e is a $(k+1)$-simplex of Π such that $v \in \partial e$, let

$$C(e) \coloneqq \{c \in \Delta^*(S(v), \Pi_1|_{S(v)}) : |c| \subset |e|\}. \tag{2.1.7}$$

[4]The second constraint can always be satisfied after a refinement.

For each such e, we define a k-simplex of $\Pi_{S(v)}$, denoted $S(v) \cap e$, whose geometric realization is

$$|S(v) \cap e| := \bigcup_{c \in C(e)} c. \tag{2.1.8}$$

We say that $S(v) \cap e'$ is a face of $S(v) \cap e$ if every $c' \in C(e')$ is a face of some $c \in C(e)$, which may depend on c'. This data defines a triangulation on $S(v)$ such that if e is a simplex of Π with $v \in \partial e$,

$$|S(v) \cap e| = |S(v)| \cap |e|. \tag{2.1.9}$$

See Figure 4 for a picture of this triangulation.

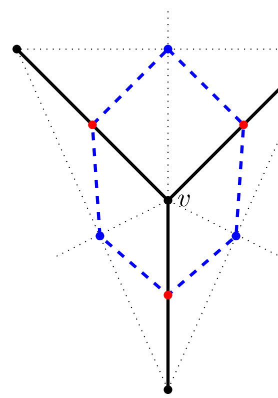


FIGURE 4. The triangulation $\Pi_{S(v)}$ constructed in Definition 2.1.6. The black vertices and solid black edges are the original simplices in Π. The remaining edges are added in the barycentric subdivision Π_1 of Π. The blue (dashed) edges and the red and blue vertices are the link $S(v)$ of v in Π_1. To define $\Pi_{S(v)}$, we keep the red vertices as 0-simplices, but for 1-simplices, the blue vertices are merged with their adjacent edges. Thus $\Pi_{S(v)}$ has three 0-simplices and three 1-simplices, and each k-simplex is the intersection of a $(k+1)$-simplex of Π with $S(v)$

From now on, the triangulation on $S(v)$ is assumed to be $\Pi_{S(v)}$ unless stated otherwise.

Definition 2.1.10. Let $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$. For any $v \in \Delta^0(M)$, let

$$Y_v(P, \xi) := \{S(v) \cap e \mid e \in Y'_v(P, \xi)\}, \tag{2.1.11}$$

which is a subcomplex of $S(v)$. The *GDS sign* [**FH16b**, §4] is

$$\sigma(v, (P, \xi)) := (-1)^{1+\chi(|Y_v(P,\xi)|)}. \tag{2.1.12}$$

Here χ denotes the Euler characteristic.

Let U_v denote the operator on $\mathcal{H}$ defined by $U_v(\psi)(P,\xi) \coloneqq \sigma(v,(P,\xi))A_v(\psi)$, where A_v is as in (1.4.10a). The Hamiltonian for the GDS model is

$$H_{\mathrm{GDS}} \coloneqq \sum_{v\in\Delta^0(M)} \widetilde{H}_v + \sum_{f\in\Delta^2(M)} H_f, \tag{2.1.13}$$

where H_f is as in (1.4.11a) and

$$\widetilde{H}_v = \frac{1-U_v}{2}. \tag{2.1.14}$$

As for the toric code, we call $\widetilde{H}_v$ a *vertex operator* and H_f a *face operator*.

Remark 2.1.15. In our analysis of the GDS model, we will need to make use of the *dual cell complex* $\Pi^\vee$ to the specified triangulation Π, a CW complex on M with several nice properties.

- $\Pi^\vee$ comes with data of a bijection $(\cdot)^\vee\colon \Delta^k(M,\Pi)\to\Delta^{n-1-k}(M,\Pi^\vee)$, sending a simplex to its *dual cell*, and such that if $e\in\partial f$, then $f^\vee\in\partial e^\vee$, and conversely.
- The map $(\cdot)^\vee$ induces a chain map on the cellular chain complexes of Π and $\Pi^\vee$ which induces Poincaré duality for the cohomology of M with $\mathbb{Z}/2$ coefficients.
- Each cell in $\Pi^\vee$ is a union of cells of the barycentric subdivision Π_1 of Π. (One might think of Π_1 as a refinement of $\Pi^\vee$; though this is not strictly true, as $\Pi^\vee$ might not come from a triangulation, it is a useful piece of intuition.) In particular, $\Pi^\vee$ is a regular CW complex, meaning the closure of each cell is contractible.

This complex is unique up to equivalence of CW complexes. Proofs of these facts follow from the results in [**Hud69**, §1.6].

We will also denote $((\cdot)^\vee)^{-1}$ by $(\cdot)^\vee$, but since we do not confuse Π and $\Pi^\vee$, the meaning will be clear from context. If S is a set of cells, we write $S^\vee \coloneqq \{e^\vee \mid e\in S\}$.

Remark 2.1.16. Freedman-Hastings [**FH16b**] study a dual version of the GDS model, in that our model for M and Π corresponds to their model for M and $\Pi^\vee$. Here we compare the two setups.

Let $(P,\xi)\in\mathrm{Bun}_{\mathbb{Z}/2}(M^1,M^0)$, which defines a function $\mathrm{spin}_{(P,\xi)}\colon\Delta^1(M,\Pi)\to\mathbb{Z}/2$ as in Definition 1.4.8; we also let $\mathrm{spin}_{(P,\xi)}$ denote the function $\Delta^{n-2}(M,\Pi^\vee)\to\mathbb{Z}/2$ defined by precomposing with $(\cdot)^\vee$.

For any $v\in\Delta^0(M,\Pi)$, let

$$T(v,(P,\xi)) \coloneqq \mathrm{spin}_{(P,\xi)}^{-1}(0)\cap\partial v^\vee, \tag{2.1.17}$$

which is a closed union of cells of $\Pi^\vee$.

The GDS sign as defined by Freedman-Hastings [**FH16b**, §4] is

$$\sigma'(v,(P,\xi)) \coloneqq (-1)^{1+\chi(T(v,(P,\xi)))}. \tag{2.1.18}$$

Let $e \in \overline{\mathrm{St}}(0)(v)$. Unwinding the definitions, $e \cap S(v) \in Y_v(P,\xi)$ if and only if $e^\vee$ is a cell of $T(v,(P,\xi))$, so the number of simplices in $Y_v(P,\xi)$ equals the number of cells in $T(v,(P,\xi))$. Since both $T(v,(P,\xi))$ and $Y_v(P,\xi)$ are closed subsets of M that are unions of cells, their Euler characteristics are equal, so $\sigma = \sigma'$. This means there is an isomorphism between the state spaces of the model we define above and the model as defined by Freedman-Hastings, and this isomorphism intertwines their Hamiltonians, so on any closed $(n-1)$-manifold, the spaces of ground states of these two models are isomorphic.

Next, we prove analogues of Proposition 1.4.16 for the GDS model. In view of Remark 2.1.16, these also follow from results of Freedman-Hastings [**FH16b**, Lemmas 4.1, 4.2], but are proven in a different way.

Lemma 2.1.19. *The Hamiltonian H_{GDS} is self-adjoint, and* $\mathrm{Spec}(H_{\mathrm{GDS}}) \subset \mathbb{Z}_{\geq 0}$.

Proof. The first part is true because the Hamiltonian is a sum of real symmetric matrices in a basis of δ-functions, just as in the proof of Proposition 1.4.16. For the second part, since the eigenvalues of A_v and B_f lie in $\{\pm 1\}$ and σ is valued in $\{\pm 1\}$, then the eigenvalues of H_f and $\widetilde{H}_v$ lie in $\{0,1\}$. $\square$

Unlike for the toric code, it is not true that 0 is always an eigenvalue. Theorem 2.3.19 and Corollary 2.4.6 together imply this happens for $M = \mathbb{CP}^{2k}$.

Lemma 2.1.20. *All face operators commute, and all face operators commute with all vertex operators. After restricting to the intersection of the kernels of the face operators, $[U_{v_1}, U_{v_2}] = 0$ and hence all vertex operators commute when restricted to that intersection.*

Proof. The face operators are the same as in the toric code, hence commute by Proposition 1.4.16. Operators corresponding to simplices not in each others' closed stars commute. Therefore we have two things left to prove:

(1) Given a 2-simplex f and a 0-simplex $v \in \partial f$, $[H_f, \widetilde{H}_v] = 0$.

(2) Given a 1-simplex e and two 0-simplices $v_1, v_2 \in \partial e$, $[U_{v_1}, U_{v_2}] = 0$ when restricted to $\bigcap_{f\in\Delta^2(M)} H_f$.

For part (1): since the GDS sign factors out of $[B_f, U_v]$, then $[B_f, U_v] = \pm[B_f, A_v] = 0$ by Proposition 1.4.16, and therefore $[H_f, \widetilde{H}_v] = 0$.

For part (2), choose $\psi \in \mathcal{H}$ such that $H_f\psi = 0$ for all 2-simplices f, and choose $(P,\xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$. Since B_f acts by multiplication by the holonomy of P around ∂f, then $\psi(P,\xi) = 0$ unless $\mathrm{Hol}_P(f) = 0$ for all f; equivalently, P must extend to all of M.[5] (This extension is necessarily unique up to isomorphism.) If this is the case,

$$
\begin{aligned}
[U_{v_1}, U_{v_2}]\psi(P,\xi) = \quad & \sigma(v_2,(P,\xi+\delta_{v_1}))\sigma(v_1,(P,\xi))\psi(P,\xi+\delta_{v_1}+\delta_{v_2}) \\
& - \sigma(v_1,(P,\xi+\delta_{v_2}))\sigma(v_2,(P,\xi))\psi(P,\xi+\delta_{v_1}+\delta_{v_2}),
\end{aligned} \tag{2.1.21}
$$

so it suffices to show that if $(P,\xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$,

$$\sigma(v_2,(P,\xi+\delta_{v_1}))\sigma(v_1,(P,\xi)) = \sigma(v_1,(P,\xi+\delta_{v_2}))\sigma(v_2,(P,\xi)). \tag{2.1.22}$$

Tracing through the definition of the GDS sign, this is equivalent to

$$\chi(|Y_{v_2}(P,\xi+\delta_{v_1})|) + \chi(|Y_{v_1}(P,\xi)|) \underset{\text{mod } 2}{\equiv} \chi(|Y_{v_1}(P,\xi+\delta_{v_2})|) + \chi(|Y_{v_2}(P,\xi)|). \tag{2.1.23}$$

Suppose $\mathrm{spin}_{(P,\xi)}(e) = 0$. For $i = 1, 2$, let $A(v_i)$ denote the set of simplices in $Y_{v_i}(P,\xi)$ contained in the closure of a simplex in $Y_{v_i}(P,\xi)$ whose closure also contains $S(v_i) \cap e$. Let $B(v_i) \coloneqq Y_{v_i}(P,\xi) \setminus A(v_i)$. Then

$$\chi(|Y_{v_2}(P,\xi+\delta_{v_1})|) + \chi(|Y_{v_1}(P,\xi)|) \underset{\text{mod } 2}{\equiv} \#(A(v_1) \amalg B(v_1) \amalg B(v_2)) \tag{2.1.24a}$$

$$\chi(|Y_{v_1}(P,\xi+\delta_{v_2})|) + \chi(|Y_{v_2}(P,\xi)|) \underset{\text{mod } 2}{\equiv} \#(A(v_2) \amalg B(v_2) \amalg B(v_1)). \tag{2.1.24b}$$

It therefore suffices to prove that $\#A(v_1) = \#A(v_2)$. Let c_1 be a 1-simplex in $A(v_1)$. Since 2-simplices are triangles, there exists a unique 1-simplex c_2 whose closure contains v_2 and such that there is a 2-simplex f with $\partial f = c_1 + c_2 + e$. By assumption, $\mathrm{spin}_{(P,\xi)}(e) = \mathrm{spin}_{(P,\xi)}(c_1) = 0$, and since the holonomy of P around ∂f vanishes, $\mathrm{spin}_{(P,\xi)}(c_2) = 0$ too. Similarly, suppose c_1' and c_2' are 1-simplices such that v_1 is a face of c_1', v_2 is a face of c_2', $\mathrm{spin}_{(P,\xi)}(c_1') = 1$, and there is a 2-simplex f' with $\partial f' = c_1' + c_2' + e$; then $\mathrm{spin}_{(P,\xi)}(c_2') = 1$ too. This argument is obviously symmetric in v_1 and v_2.

The case $\mathrm{spin}_{(P,\xi)}(e) = 1$ is analogous. □

Remark 2.1.25. The ideas that go into the GDS model still make sense when one generalizes to smooth manifolds with regular CW structures, rather than smooth triangulations, but Lemma 2.1.20 does not generalize. See Figure 5 for a counterexample.

If one lets $n = 3$ and passes to the dual CW structure as in Remark 2.1.15, this recovers a fact known to condensed-matter theorists: the double semion model on a surface can be formulated on a hexagonal lattice (or more generally a trivalent lattice), but has an ambiguity when placed on a square lattice [**FH16b**, §2]. This is because the dual CW structure to a trivalent lattice has triangular 2-cells, but the dual of a

[5] We will return to this point in §2.3.2.

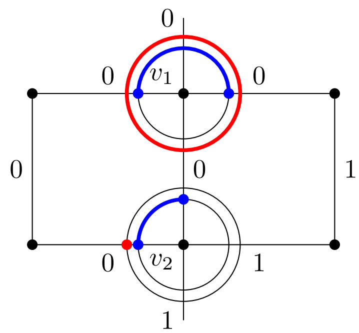


FIGURE 5. Lemma 2.1.20 does not generalize from triangulations to CW structures. The straight lines in this figure depict a neighborhood on a smooth surface Σ with a CW structure. Choose $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(\Sigma^1, \Sigma^0)$ such that the number on each pictured 1-cell e is $\mathrm{spin}_{(P,\xi)}(e)$. The circles around the 0-cells v_1 and v_2 represent two copies of each of the links $S(v_1)$ and $S(v_2)$. The red region (shaded portions of the outer circles) is $|Y_{v_1}(P,\xi)| \amalg |Y_{v_2}(P, \xi + \delta_{v_1})|$, and the blue region (shaded portions of the inner circles) is $|Y_{v_2}(P,\xi)| \amalg |Y_{v_1}(P, \xi + \delta_{v_2})|$. By inspection, the Euler characteristics of these two regions are not equal mod 2, so (2.1.23) does not hold in this setting, and therefore Lemma 2.1.20 also does not apply to this CW structure: $\widetilde{H}_{v_1}$ and $\widetilde{H}_{v_2}$ do not commute even when restricted to $\bigcap_f H_f$.

tetravalent lattice does not. For general n, this obstruction is encoded in the genericity assumption placed on the CW structure in Freedman-Hastings' construction [**FH16b**, §4]; in our model this corresponds to the restriction to smooth triangulations.

Lemma 2.1.26. *The face operators are projectors. The operator U_v has order 2, and hence $\widetilde{H}_v$ is a projector.*

PROOF. The face operators are the same as in the toric code, hence are projectors by Proposition 1.4.16. For U_v, choose a 0-simplex v, $\psi \in \mathcal{H}$, and $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$; then,

$$U_v^2 \psi(P,\xi) = \sigma(v, (P, \xi + \delta_v)\sigma(v, (P,\xi))\psi(P,\xi) = (-1)^{\chi(|Y_v(P,\xi+\delta_v)|) + \chi(|Y_v(P,\xi)|)}\psi(P,\xi). \tag{2.1.27}$$

Unwinding the definition of Y_v, and using that $\chi(S(v)) \equiv 0 \bmod 2$, $\chi(|Y_v(P, \xi+\delta_v)|) + \chi(|Y_v(P,\xi)|)$ is equal mod 2 to the number of simplices e in $S(v)$ such that $\overline{e}$ contains a 0-simplex on which ξ extends and a 0-simplex on which $\xi + \delta_v$ extends (equivalently, on which ξ does not extend). Let Q be the set of such e.

Endow $S(v)$ with the Poincaré dual CW structure $\Pi^\vee_{S(v)}$ to the triangulation $\Pi_{S(v)}$, as in Remark 2.1.15. Let $R \subset \Pi_{S(v)}$ be the set of 1-simplices on which ξ extends; then, $|R^\vee|$ is a topological submanifold (with boundary) of $S(v)$, and $\partial |R^\vee| = |Q^\vee|$. Hence $\chi(|Q^\vee|) \equiv 0 \bmod 2$; since $Q^\vee$ is a subcomplex of $\Pi^\vee_{S(v)}$, this means $Q^\vee$ has an even number of cells, so Q has an even number of simplices. Thus $\chi(|Y_v(P,\xi+\delta_v)|) + \chi(|Y_v(P,\xi)|) \equiv 0 \bmod 2$, and this suffices by (2.1.27). $\square$

There are a few other equivalent ways to define the GDS sign. We record one which we will use later.

Proposition 2.1.28. *Let $(P,\xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$ and $v \in \Delta^0(M)$, and let N_v be the set of simplices c of M with $v \in \partial c$. If $Z_v(P,\xi) \subset N_v$ denotes the subset of simplices c such that either (1) c is a 1-simplex and $\mathrm{spin}_{(P,\xi)}(c) = 1$, or (2) there is a 1-simplex $e \in \partial c$ with $\mathrm{spin}_{(P,\xi)}(e) = 1$, then $(-1)^{1+\#Z_v(P,\xi)} = \sigma(v,(P,\xi))$.*

Proof. It suffices to show $\#Z_v(P,\xi) \equiv \#Y_v(P,\xi) \bmod 2$. If $W_v(P,\xi)$ denotes the subset of N_v consisting of simplices c such that either (1) c is a 1-simplex and $\mathrm{spin}_{(P,\xi)}(c) = 0$, or (2) $\mathrm{spin}_{(P,\xi)}(e) = 0$ for all $e \in \Delta^1(\partial c)$, then the map $c \mapsto c \cap S(v)$ for $c \in N_v$ restricts to a bijection from $W_v(P,\xi)$ to $Y_v(P,\xi)$.

By definition, $Z_v(P,\xi)$ is the complement of $W_v(P,\xi)$ inside N_v. Since $N_v^\vee = \partial v^\vee$ and $\chi(|\partial v^\vee|)$ is even, then $\#N_v$ is even and

$$\#Z_v(P,\xi) + \#Y_v(P,\xi) = \#Z_v(P,\xi) + \#W_v(P,\xi) = \#N_v \equiv 0 \bmod 2. \tag{2.1.29}$$

□

2.2. Gauge-gravity TFTs

As part of our goal of studying the low-energy behavior of the GDS model, we would like a description in terms of a TFT whose state spaces we can compute relatively easily. The answer comes to us as one of a class of TFTs, called $\mathbb{Z}/2$-gauge-gravity theories; these TFTs are slight generalizations of $\mathbb{Z}/2$-Dijkgraaf-Witten theories [**DW90, FQ93**], in which Stiefel-Whitney classes of the underlying manifold can enter the Lagrangian action. Theories of this sort have also been considered by Kapustin [**Kap14a, Kap14b**], Wen [**Wen15, Wen17**], and Lan-Kong-Wen [**LKW18**], though not in this generality.

As in the construction of Dijkgraaf-Witten theories, we will construct the gauge-gravity theories in two steps. First, we will construct invertible theories for unoriented manifolds equipped with a principal $\mathbb{Z}/2$-bundle; these are the classical theories, and are examples of Turaev's homotopy quantum field theories with target $B\mathbb{Z}/2$ [**Tur10**] (sometimes also called equivariant TFTs [**SW18**]). Then, we will sum over principal $\mathbb{Z}/2$-bundles in a process called orbifoldization, producing what are called the quantum theories [**FQ93, FHLT10**] or the orbifold theories [**SW18**].

2.2.1. Construction of the classical $\mathbb{Z}/2$-gauge-gravity theories.

Recall that a topological field theory $Z\colon \mathsf{Bord}^{\xi}_{n,n-1}(X) \to \mathsf{Vect}_{\mathbb{C}}$ is *invertible* if it factors through the subgroupoid $\mathsf{Line}_{\mathbb{C}} \hookrightarrow \mathsf{Vect}_{\mathbb{C}}$ of complex lines and nonzero homomorphisms. This means, for example, that all partition functions are nonzero and all state spaces are one-dimensional.

Theorem 2.2.1. *Let $\beta \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$. Then there is an invertible TFT $Z^{\mathrm{cl}}_\beta\colon \mathsf{Bord}^{\mathrm{O}}_{n,n-1}(B\mathbb{Z}/2) \to \mathsf{Vect}_{\mathbb{C}}$ of n-manifolds equipped with a principal $\mathbb{Z}/2$-bundle, unique up to isomorphism, such that for any* *closed n-manifold M and principal $\mathbb{Z}/2$-bundle $P \to M$,*

$$Z^{\mathrm{cl}}_\beta(M, P) = (-1)^{\langle \beta(M,P), [M]\rangle}, \tag{2.2.2}$$

where $\beta(M, P)$ denotes the pullback of β under a map $M \to B\mathrm{O}_n \times B\mathbb{Z}/2$ classifying TM and P.[6]

Proof. The assignment (2.2.2) is a $\{\pm 1\}$-valued bordism invariant of manifolds equipped with a principal $\mathbb{Z}/2$-bundle. Composing with the unique nonzero map $\{\pm 1\} \hookrightarrow \mathrm{U}_1$, we obtain (2.2.2) as a U_1-valued bordism invariant. Assume for now that the bordism group $\Omega^{\mathrm{O}}_{n-1}(B\mathbb{Z}/2)$ of $(n-1)$-dimensional manifolds with a principal $\mathbb{Z}/2$-bundle is finitely generated; using this assumption, Yonekura [**Yon19**, Theorems 4.3 and 4.4] constructs an invertible TFT valued in $\mathsf{Line}_{\mathbb{C}}$ whose partition function recovers the bordism invariant (2.2.2), and proves that it is unique up to isomorphism.

Now we show $\Omega^{\mathrm{O}}_{n-1}(B\mathbb{Z}/2)$ is finitely generated. By Theorem 1.1.35,

$$\Omega^{\mathrm{O}}_{n-1}(B\mathbb{Z}/2) \cong \bigoplus_{i+j=n-1} H_i(B\mathbb{Z}/2; \Omega^{\mathrm{O}}_j). \tag{2.2.3}$$

Theorem 1.1.35 shows Ω^{O}_j is finitely generated, and $B\mathbb{Z}/2$ has a CW structure with finitely many cells in each dimension, so the right-hand side of (2.2.3) is also finitely generated. $\square$

We call Z^{cl}_β the *classical $\mathbb{Z}/2$-gauge-gravity theory* for β, and call β the *Lagrangian* for the theory.

Remark 2.2.4. The name "gauge-gravity" refers to the fact that the Lagrangian β can have terms depending both on the principal $\mathbb{Z}/2$-bundle (a gauge field) and characteristic classes of the underlying manifold (which, due to the relationship between characteristic classes and curvature, are sometimes called gravitational terms). This idea also appears for the anomaly TFTs in [**STY18, GEM19**], which are similar to the classical gauge-gravity theories considered in this paper.

Remark 2.2.5. It is also possible to describe Z^{cl}_β homotopically, following the Freed-Hopkins-Teleman approach to invertible TFTs [**FH16a**] that we described in §1.3: we saw there that an invertible TFT $Z^{\mathrm{cl}}\colon \mathsf{Bord}^{\mathrm{O}}_{n,n-1}(B\mathbb{Z}/2) \to \mathsf{Line}_{\mathbb{C}}$ determines and is determined up to isomorphism by the homotopy class of the map

$$|Z^{\mathrm{cl}}|\colon |\mathsf{Bord}^{\mathrm{O}}_{n,n-1}(B\mathbb{Z}/2)| \to |\mathsf{Line}_{\mathbb{C}}| \tag{2.2.6}$$

it induces on classifying spectra.

[6] The classifying map is unique up to homotopy, so $\beta(M, P)$ does not depend on this choice.

If E is a spectrum, let $\tau_{m:n}E$ denote the truncation of E to a spectrum with homotopy groups only in degrees between m and n, inclusive. Then there are weak equivalences

- $|\mathsf{Bord}^{\mathrm{O}}_{n,n-1}(B\mathbb{Z}/2)| \simeq \tau_{0:1}(\Sigma MTO_n \wedge (B\mathbb{Z}/2)_+)$ [**GMTW09, Ngu17**],[7] and
- $|\mathsf{Line}_{\mathbb{C}}| \simeq \Sigma H\mathbb{C}^\times$.

Therefore an isomorphism class of invertible n-dimensional TFTs for manifolds with a principal $\mathbb{Z}/2$-bundle is determined by an element of

$$[\tau_{0:1}(\Sigma MTO_n \wedge (B\mathbb{Z}/2)_+), \Sigma H\mathbb{C}^\times] \cong H^0(MTO_n \wedge (B\mathbb{Z}/2)_+; \mathbb{C}^\times), \tag{2.2.7}$$

and $\beta \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$ yields such an element through the mod 2 Thom isomorphism followed by the map induced on cohomology by $\mathbb{Z}/2 \cong \{\pm 1\} \hookrightarrow \mathbb{C}^\times$. Thus it defines an invertible TFT $(Z^{\mathrm{cl}}_\beta)'$ up to isomorphism. Tracing through the Pontrjagin-Thom construction, one can prove that its partition functions agree with those in (2.2.2), and hence by Yonekura's uniqueness result [**Yon19**, Theorem 4.4], $(Z^{\mathrm{cl}}_\beta)' \cong Z^{\mathrm{cl}}_\beta$.

This approach readily generalizes to extended invertible TFTs, as in [**SP17**], and the classical gauge-gravity TFTs can be realized as fully extended TFTs valued in n-algebras, as in [**FHLT10**, §8], or n-vector spaces, using the calculation of the classifying spectrum of the n-category of n-vector spaces in [**SP17**, §7.4].

The partition functions of the classical gauge-gravity TFT for β resemble those of classical Dijkgraaf-Witten theory [**DW90, FQ93**] for the gauge group $\mathbb{Z}/2$, though the Lagrangians of the former can also contain Stiefel-Whitney classes. If β factors through the inclusion $H^n(B\mathbb{Z}/2; \mathbb{Z}/2) \hookrightarrow H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$, then Z^{cl}_β is isomorphic to a classical $\mathbb{Z}/2$-Dijkgraaf-Witten theory.

If $\gamma \in H^n(B\mathbb{Z}/2; \mathbb{R}/\mathbb{Z})$, we let $\mathrm{DW}^{\mathrm{cl}}_\gamma$ denote classical $\mathbb{Z}/2$-Dijkgraaf-Witten theory with Lagrangian γ.

Proposition 2.2.8. *Let $f\colon \mathbb{Z}/2 \hookrightarrow \mathbb{R}/\mathbb{Z}$ denote the map sending $1 \mapsto 1/2$, as well as the map $f\colon H^*(X; \mathbb{Z}/2) \to H^*(X; \mathbb{R}/\mathbb{Z})$ it induces on cohomology. Suppose β contains no Stiefel-Whitney terms, i.e. β factors through $H^n(B\mathbb{Z}/2; \mathbb{Z}/2) \hookrightarrow H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$. Then, as TFTs of oriented manifolds equipped with principal $\mathbb{Z}/2$-bundles, $Z^{\mathrm{cl}}_\beta \cong \mathrm{DW}^{\mathrm{cl}}_{f(\beta)}$.*

Proof. Let M be a closed, oriented n-manifold, $P \to M$ be a principal $\mathbb{Z}/2$-bundle, and β be as in the proposition statement. Let $\phi\colon M \to B\mathbb{Z}/2$ be a classifying map for P. Let $[M]_{\mathbb{Z}}$, resp. $[M]_{\mathbb{Z}/2}$, denote the fundamental class of M in integral, resp. $\mathbb{Z}/2$, homology.

The partition function of classical $\mathbb{Z}/2$-Dijkgraaf-Witten theory with Lagrangian $f(\beta)$ is $\mathrm{DW}^{\mathrm{cl}}_{f(\beta)}(M, P) = e^{2\pi i \langle (\phi^*(f(\beta)), [M]_{\mathbb{Z}}\rangle}$ [**FQ93**, Theorem 1.7]. Naturality of the cap product under change of coefficients implies

[7] This fact has been proven or sketched in several additional ways: see also [**Aya09, Lur09b, BM14, AF17, SP17**].

$f(\langle x, [M]_{\mathbb{Z}/2}\rangle) = \langle f(x), [M]_{\mathbb{Z}}\rangle$ for any $x \in H^n(M;\mathbb{Z}/2)$, and naturality of the change-of-coefficients map on cohomology implies that $\phi^*(f(\beta)) = f(\phi^*(\beta))$, so $f(\langle \phi^*\beta, [M]_{\mathbb{Z}/2}\rangle) = \langle \phi^*(f(\beta)), [M]_{\mathbb{Z}}\rangle$. If $a \in \mathbb{Z}/2$, $(-1)^a = \mathrm{e}^{2\pi i f(a)}$, so

$$Z_\beta(M,P) = (-1)^{\langle \phi^*\beta, [M]_{\mathbb{Z}/2}\rangle} = \mathrm{e}^{2\pi i\langle \phi^*(f(\beta)), [M]_{\mathbb{Z}}\rangle} = \mathrm{DW}^{\mathrm{cl}}_{f(\beta)}(M,P). \tag{2.2.9}$$

Since the partition functions for these theories are identical, then by [**Yon19**, Theorem 4.4], $Z_\beta \cong \mathrm{DW}^{\mathrm{cl}}_{f(\beta)}$. $\square$

Remark 2.2.10. One takeaway from Proposition 2.2.8 is that when β contains no Stiefel-Whitney terms, Z^{cl}_β is an extension of $\mathrm{DW}^{\mathrm{cl}}_{f(\beta)}$ to unoriented manifolds. Such extensions are studied in detail by Young [**You20**] in both the classical and quantum settings, and are examples of Minkyu Kim's generalized Dijkgraaf-Witten theories [**Kim18**].

Remark 2.2.11. These classical gauge-gravity theories are examples of homotopy quantum field theories (HQFTs) with target space $B\mathbb{Z}/2$, and in this setting they resemble primitive cohomological HQFTs [**Tur10**, §I.2.1]; again the difference is whether the cohomology class can contain Stiefel-Whitney terms. The construction of primitive cohomological HQFTs is quite direct, and it seems likely that the classical gauge-gravity theories can be constructed in a similar way.

Lemma 2.2.12. *Let $\gamma \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$ be a cohomology class which vanishes when pulled back to all closed n-manifolds via a classifying map for the tangent bundle and any principal $\mathbb{Z}/2$-bundle. Then, $Z^{\mathrm{cl}}_\beta \cong Z^{\mathrm{cl}}_{\beta+\gamma}$.*

Proof. By (2.2.2), $Z_\beta(M) = Z_{\beta+\gamma}(M)$ for all closed n-manifolds M with a principal $\mathbb{Z}/2$-bundle. We have seen that invertible TFTs of manifolds with a principal $\mathbb{Z}/2$-bundle are determined up to isomorphism by their partition functions, so $Z_\beta \cong Z_{\beta+\gamma}$. $\square$

For example, in dimension 3, $w_1^2\alpha = w_2\alpha$ on all 3-manifolds, so $Z^{\mathrm{cl}}_{w_1^2\alpha} \cong Z^{\mathrm{cl}}_{w_2\alpha}$.

Lemma 2.2.13. *If n is odd, the map $f\colon H^n(B\mathbb{Z}/2;\mathbb{Z}/2) \to H^n(B\mathbb{Z}/2;\mathbb{R}/\mathbb{Z})$ is surjective.*

Proof. Associated to the short exact sequence $0 \to \mathbb{Z}/2 \xrightarrow{f} \mathbb{R}/\mathbb{Z} \xrightarrow{2} \mathbb{R}/\mathbb{Z} \to 0$, there is a long exact sequence in cohomology:

(2.2.14)

$$H^n(B\mathbb{Z}/2;\mathbb{Z}/2) \xrightarrow{f} H^n(B\mathbb{Z}/2;\mathbb{R}/\mathbb{Z}) \xrightarrow{g} H^n(B\mathbb{Z}/2;\mathbb{R}/\mathbb{Z}) \xrightarrow{h} H^{n+1}(B\mathbb{Z}/2;\mathbb{Z}/2) \longrightarrow H^{n+1}(B\mathbb{Z}/2;\mathbb{R}/\mathbb{Z}).$$

Since $H^{n+1}(B\mathbb{Z}/2;\mathbb{R}/\mathbb{Z}) = 0$, h is surjective. Since n is odd, both $H^n(B\mathbb{Z}/2;\mathbb{R}/\mathbb{Z})$ and $H^{n+1}(B\mathbb{Z}/2;\mathbb{Z}/2)$ are isomorphic to $\mathbb{Z}/2$, so h is a surjective map $\mathbb{Z}/2 \to \mathbb{Z}/2$, hence an isomorphism. Thus $g = 0$, so f is surjective as desired. □

Corollary 2.2.15. *If n is odd, every classical $\mathbb{Z}/2$-Dijkgraaf-Witten theory is isomorphic to Z_β^{cl} for some $\beta \in H^n(B\mathbb{Z}/2;\mathbb{Z}/2) \hookrightarrow H^n(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$.*

Proof. By Lemma 2.2.13, when n is odd, the map $f\colon H^n(B\mathbb{Z}/2;\mathbb{Z}/2) \to H^n(B\mathbb{Z}/2;\mathbb{R}/\mathbb{Z})$ is surjective; then the result follows from Proposition 2.2.8. □

2.2.2. Discussion of the quantum theories. We construct the quantum theory Z_β using the finite path integral approach of [**FHLT10**, §3]; see also [**Mor15, Tro16**] for a more detailed account and [**SW18**] for a related construction. This process is also known as *orbifolding*, and the quantum theory Z_β is sometimes called the *orbifold theory* for Z_β^{cl}.

Let Gpd denote the category of spans of essentially finite groupoids: the objects of Gpd are essentially finite groupoids, and a morphism from X_1 to X_2 is data of a essentially finite groupoid Y and functors $p_1\colon Y \to X_1$ and $p_2\colon Y \to X_2$, considered up to equivalence of (Y, p_1, p_2). Let $\mathsf{Gpd}(\mathsf{Vect}_\mathbb{C})$ denote the category whose objects are pairs (X, V), where X is an essentially finite groupoid and $V \to X$ is a complex vector bundle,[8] and whose morphisms are equivalence classes of spans

$$\begin{tikzcd} & Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & & X_2 \end{tikzcd} \tag{2.2.16}$$

together with data of vector bundles $V_i \to X_i$ and $W \to Y$ and morphisms $\phi_i\colon p_i^* V_i \to W$ for $i = 1, 2$. For any $y \in Y$, this morphism determines a linear map $\varphi(y)\colon V_1(p_1(y)) \to V_2(p_2(y))$ by a push-pull construction. Disjoint union of groupoids defines a symmetric monoidal structure on $\mathsf{Gpd}(\mathsf{Vect}_\mathbb{C})$.

We next define the "quantization" functor $\Sigma\colon \mathsf{Gpd}(\mathsf{Vect}_\mathbb{C}) \to \mathsf{Vect}_\mathbb{C}$, which on to an object assigns

$$\Sigma\colon (X, V) \mapsto \Gamma(V) \coloneqq \varinjlim_{x \in X} V(x), \tag{2.2.17}$$

i.e. regard V as a $\mathsf{Vect}_\mathbb{C}$-valued diagram indexed by the category X, and take the colimit of this diagram. Given a morphism (Y, W, ϕ_1, ϕ_2) as above, the maps $\varphi(y)$ for $y \in Y$ pass to the colimit to define a map

$$\widetilde{\varphi}\colon \pi_0 Y \to \mathrm{Hom}(\Gamma(X_1, V_1), \Gamma(X_2, V_2)). \tag{2.2.18}$$

[8] A (complex) vector bundle over a groupoid $\mathcal{G}$, denoted $V \to \mathcal{G}$, is a functor $V\colon \mathcal{G} \to \mathsf{Vect}_\mathbb{C}$, and its space of sections is $\varinjlim V$. We will always assume these vector bundles are finite-dimensional, meaning they factor through the full subcategory of finite-dimensional vector spaces.

Then, Σ assigns to this morphism the linear map

$$\Sigma(Y, W) \coloneqq \sum_{[y]\in\pi_0 Y} \frac{\widetilde{\varphi}(y)}{|\mathrm{Aut}(y)|} \in \mathrm{Hom}(\Gamma(X_1, V_1), \Gamma(X_2, V_2)). \tag{2.2.19}$$

This functor is symmetric monoidal [**Tro16**, Theorem 5.1].

Given a TFT $Z^{\mathrm{cl}}\colon \mathsf{Bord}^{\mathrm{O}}_{n,n-1}(B\mathbb{Z}/2) \to \mathsf{Vect}_{\mathbb{C}}$, the functor $F_{Z^{\mathrm{cl}}}\colon \mathsf{Bord}^{\mathrm{O}}_{n,n-1} \to \mathsf{Gpd}(\mathsf{Vect}_{\mathbb{C}})$ sending

$$F_{Z^{\mathrm{cl}}}\colon M \mapsto \big(\mathrm{Bun}_{\mathbb{Z}/2}(M), P \mapsto Z^{\mathrm{cl}}(M, P)\big) \tag{2.2.20}$$

is also symmetric monoidal [**SW19**, Theorem 3.9], and therefore the composition

$$Z\colon \mathsf{Bord}^{\mathrm{O}}_{n,n-1} \xrightarrow{F_{Z^{\mathrm{cl}}}} \mathsf{Gpd}(\mathsf{Vect}_{\mathbb{C}}) \xrightarrow{\Sigma} \mathsf{Vect}_{\mathbb{C}} \tag{2.2.21}$$

is symmetric monoidal, i.e. a (nonextended) TFT of unoriented manifolds.

Definition 2.2.22. Given a TFT $Z^{\mathrm{cl}}\colon \mathsf{Bord}^{\mathrm{O}}_{n,n-1}(B\mathbb{Z}/2) \to \mathsf{Vect}_{\mathbb{C}}$, the TFT Z in (2.2.21) above is called the *quantum theory* associated to Z^{cl}. In particular, we denote the quantum theory associated to Z^{cl}_β by Z_β, and call it the *(quantum) gauge-gravity theory* for β. In this case we call β the *Lagrangian* of the theory.

Proposition 2.2.23 ([**SW19**, Corollary 4.4])**.**

(1) Let M be a closed n-manifold. Then, the partition function $Z_\beta(M)$ is

$$Z_\beta(M) = \sum_{[P]\in\pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)} \frac{(-1)^{\langle\beta(P),[M]\rangle}}{|\mathrm{Aut}(P)|}. \tag{2.2.24}$$

(2) Let N be a closed $(n-1)$-manifold. Then, define a line bundle $L_\beta \to \mathrm{Bun}_{\mathbb{Z}/2}(N)$ which

- *assigns $\mathbb{C}$ to every object, and*
- *assigns to an automorphism $\phi \in \mathrm{Aut}(P)$ multiplication by $Z^{\mathrm{cl}}_\beta(S^1 \times N, P_\phi)$.*

Then the state space of N is $Z_\beta(N) \cong \Gamma(L_\beta)$.

Here $P_\phi \to S^1 \times N$ denotes the *mapping torus* of ϕ, i.e. the quotient of $[0,1] \times P$ by $(0,x) \sim (1, \phi(x))$. We sketch the proof; the details can be found in [**SW19**, §§3,4].

Proof. First, part (1). The partition function for M is $Z_\beta(M\colon \varnothing \to \varnothing)$. To this bordism, $F_{Z^{\mathrm{cl}}_\beta}$ assigns a span such that for any $P \in \mathrm{Bun}_{\mathbb{Z}/2}(M)$, the induced map $\varphi(P)\colon \mathbb{C} \to \mathbb{C}$ is multiplication by the classical partition function $Z^{\mathrm{cl}}_\beta(M, P)$. Applying Σ sums this over $[P] \in \pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$, weighted by automorphisms, giving (2.2.24).

Now part (2). $F_{Z^{\mathrm{cl}}_\beta}$ sends N to a line bundle $L_N \to \mathrm{Bun}_{\mathbb{Z}/2}(N)$, which to a principal $\mathbb{Z}/2$-bundle $P \to N$ assigns the complex line $Z^{\mathrm{cl}}_\beta(N, P)$. Given a morphism, let $\mathrm{Cyl}^\phi(P) \to [0,1] \times N$ denote the *mapping cylinder* of ϕ, i.e. the space $P \times [0,1] \to N \times [0,1]$, interpreted as a bordism in which P is glued by the identity at 0 and by ϕ at 1. Then,

$$L_N(\phi) = Z^{\mathrm{cl}}_\beta([0,1] \times N, \mathrm{Cyl}^\phi(P))\colon Z^{\mathrm{cl}}_\beta(N, P) \to Z^{\mathrm{cl}}_\beta(N, P) \tag{2.2.25a}$$

$$= (\text{multiplication by } Z^{\mathrm{cl}}_\beta(S^1 \times N, P_\phi))\colon Z^{\mathrm{cl}}_\beta(N, P) \to Z^{\mathrm{cl}}_\beta(N, P). \tag{2.2.25b}$$

. Thus $L_N \to \mathrm{Bun}_{\mathbb{Z}/2}(N)$ is isomorphic to L_β from the proposition statement, so $Z_\beta(N) = \Gamma(L_\beta)$. ☐

The finite path integral approach to defining the quantum gauge-gravity theories means a few of their basic properties are formal corollaries of their counterparts in the classical case, because an isomorphism of classical theories determines an isomorphism of quantum theories.

Corollary 2.2.26. *Let $\gamma \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$ be a cohomology class which vanishes when pulled back to all closed n-manifolds via a classifying map for the tangent bundle and any principal $\mathbb{Z}/2$-bundle. Then, $Z_\beta \cong Z_{\beta+\gamma}$.*

Corollary 2.2.27. *Suppose β contains no Stiefel-Whitney terms (in the sense of Proposition 2.2.8). Then, $Z_\beta \cong \mathrm{DW}_\beta$, the quantum $\mathbb{Z}/2$-Dijkgraaf-Witten theory with Lagrangian β.*

Corollary 2.2.28. *If n is odd, every quantum $\mathbb{Z}/2$-Dijkgraaf-Witten theory is isomorphic to Z_β for some $\beta \in H^n(B\mathbb{Z}/2; \mathbb{Z}/2) \hookrightarrow H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$.*

There is a new phenomenon at this level, however: one can produce β and β' whose quantum theories are isomorphic, but whose classical theories are not.

Definition 2.2.29. Let $\beta \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2)$, so that there are coefficients $\gamma_1, \dots, \gamma_n \in H^*(B\mathrm{O}_n; \mathbb{Z}/2)$ such that

$$\beta = \gamma_n \alpha^n + \gamma_{n-1}\alpha^{n-1} + \cdots + \gamma_1 \alpha + \gamma_0, \tag{2.2.30}$$

where $\alpha \in H^1(B\mathbb{Z}/2; \mathbb{Z}/2)$ is the generator. If $w_1 \in H^1(B\mathrm{O}_n; \mathbb{Z}/2)$ denotes the first Stiefel-Whitney class, we call

$$\beta_{w_1} \coloneqq \gamma_n(\alpha + w_1)^n + \gamma_{n-1}(\alpha + w_1)^{n-1} + \cdots + \gamma_1(\alpha + w_1) + \gamma_0 \in H^n(B\mathrm{O}_n \times B\mathbb{Z}/2; \mathbb{Z}/2) \tag{2.2.31}$$

the *orientation-twisting* of β.

Proposition 2.2.32. *Let β_{w_1} be the orientation-twisting of β. Then, $Z_\beta \cong Z_{\beta_{w_1}}$.*

The idea is that replacing β with β_{w_1} corresponds to tensoring with the orientation bundle, an involution on the space of fields. Since we are summing over the fields, this does not change the path integral.

Definition 2.2.33. We define a tensor product of principal $\mathbb{Z}/2$-bundles induced from the tensor product of real line bundles. Given two principal $\mathbb{Z}/2$-bundles $P_1, P_2 \to M$, define a real line bundle $L(P_i) \to M$ for $i = 1, 2$ by $L(P_i) \coloneqq P_i \times_{\mathbb{Z}/2} \mathbb{R}$, where $\mathbb{Z}/2$ acts on $\mathbb{R}$ as $\{\pm 1\}$. The Euclidean metric on $\mathbb{R}$ induces Euclidean metrics on $L(P_1)$ and $L(P_2)$, hence also on $L(P_1) \otimes L(P_2)$; we define the *tensor product* of P_1 and P_2, denoted $P_1 \otimes P_2 \to M$, to be the unit sphere bundle in $L(P_1) \otimes L(P_2)$, which is a principal $\mathbb{Z}/2$-bundle on M.

The characteristic class of $P \otimes Q$ is $\alpha(P \otimes Q) = \alpha(P) + \alpha(Q)$.

On any manifold M, there is a canonical principal $\mathbb{Z}/2$-bundle $\mathfrak{o}_M$, called the *orientation bundle*, whose fiber at $x \in M$ is the $\mathbb{Z}/2$-torsor of orientations at x. Its characteristic class is $\alpha(\mathfrak{o}_M) = w_1(M)$.

Proof of Proposition 2.2.32. Let PM_n denote the subcategory of $\mathsf{Gpd}(\mathsf{Vect}_\mathbb{C})$ whose objects are vector bundles over groupoids of the form $\mathrm{Bun}_{\mathbb{Z}/2}(N)$ for some closed $(n-1)$-manifold N and whose morphisms are induced from the spans

$$\begin{tikzcd} & \mathrm{Bun}_{\mathbb{Z}/2}(M) \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_{\mathbb{Z}/2}(N_1) & & \mathrm{Bun}_{\mathbb{Z}/2}(N_2), \end{tikzcd} \tag{2.2.34}$$

where M is a bordism between N_1 and N_2. For any β, $F_{Z_\beta^{\mathrm{cl}}}$ lands in PM_n. To simplify notation, we will let $F_\beta \coloneqq F_{Z_\beta^{\mathrm{cl}}}$.

If M is a bordism between N_1 and N_2, $(\mathfrak{o}_M)|_{N_i} = \mathfrak{o}_{N_i}$. Thus the automorphism $-\otimes \mathfrak{o}_Y \colon \mathrm{Bun}_{\mathbb{Z}/2}(Y) \to \mathrm{Bun}_{\mathbb{Z}/2}(Y)$ induces an automorphism $\Phi\colon \mathsf{PM}_n \to \mathsf{PM}_n$ as follows.

- An object of PM_n is a functor $F\colon \mathrm{Bun}_{\mathbb{Z}/2}(N) \to \mathsf{Vect}_\mathbb{C}$ for some $(n-1)$-manifold N. Let $\Phi(F)$ be $F \circ (-\otimes \mathfrak{o}_N)\colon \mathrm{Bun}_{\mathbb{Z}/2}(N) \to \mathrm{Bun}_{\mathbb{Z}/2}(N) \to \mathsf{Vect}_\mathbb{C}$.
- A morphism $F_1 \to F_2$ of PM_n is a push-pull map induced from a span as in (2.2.34). Since $(\mathfrak{o}_M)|_{N_i} = \mathfrak{o}_{N_i}$, the arrows in (2.2.34) intertwine the actions of $-\otimes \mathfrak{o}_M$ and $-\otimes \mathfrak{o}_{N_i}$, so this span induces a morphism $\Phi(F_1) \to \Phi(F_2)$ as desired.

Thus we may consider the diagram

$$
\begin{tikzcd}
\mathsf{Bord}^{\mathrm{O}}_{n,n-1} \arrow[r, "F_\beta"] \arrow[dr, "F_{\beta_{w_1}}"'] & \mathsf{PM}_n \arrow[r, "\Sigma"] \arrow[d, "\Phi"] & \mathsf{Vect}_{\mathbb{C}} \\
& \mathsf{PM}_n, \arrow[ur, "\Sigma"'] &
\end{tikzcd}
\tag{2.2.35}
$$

where the composition along the top is Z_β and the composition along the bottom is $Z_{\beta_{w_1}}$.

It suffices to prove this diagram commutes up to natural isomorphism, which means checking its two triangles.

- The left triangle commutes (up to natural isomorphism) by design, since $\alpha(P \otimes \mathfrak{o}_M) = \alpha(P) + w_1(M)$ and in β_{w_1}, we have replaced α with $\alpha + w_1$.
- The right triangle commutes because Σ takes a diagram and evaluates its colimit, and an automorphism of the indexing category does not change the value of the colimit. Hence $\Sigma(S)$ and $(\Sigma \circ \Phi)(S)$ are isomorphic for any object S, and since Φ is compatible with morphisms in PM_n, Σ and $\Sigma \circ \Phi$ also agree on morphisms. $\square$

Example 2.2.36. The orientation twisting of α^2 is $\alpha^2 + w_1^2$. The classical theories $Z^{\mathrm{cl}}_{\alpha^2}$ and $Z^{\mathrm{cl}}_{\alpha^2+w_1^2}$ are nonisomorphic; for example, they disagree on $\mathbb{RP}^2$ with the trivial principal $\mathbb{Z}/2$-bundle. But by Proposition 2.2.32, their quantum theories are isomorphic.

Remark 2.2.37. Lu-Vishwanath [**LV16**] observe a similar phenomenon in the physics of topological phases enriched by a global $\mathbb{Z}/2$-symmetry, in which distinct phases become equivalent after gauging the $\mathbb{Z}/2$ symmetry.

2.3. Low-energy limits

In this section, we return to the lattice, and investigate the spaces of ground states of the toric code and GDS models on closed $(n-1)$-manifolds. In both cases, we extract a truncated TFT from the lattice model and show that it is isomorphic to the truncation of a $\mathbb{Z}/2$-gauge-gravity TFT.

2.3.1. Generalities.

Definition 2.3.1. Consider a lattice model which to all closed $(n-1)$-manifolds M together with some kind of lattice Π (e.g. a triangulation or a CW structure) associates a complex Hilbert space $\mathcal{H}_{M,\Pi}$ and a self-adjoint operator $H_{M,\Pi}\colon \mathcal{H}_{M,\Pi} \to \mathcal{H}_{M,\Pi}$ (respectively the state space and the Hamiltonian). In this setting, elements of $\ker(H_{M,\Pi})$ are called *ground states*. Assume that we can construct an action of $\mathrm{Diff}(M)$ on $\ker(H_{M,\Pi})$ from the data of the lattice model.

In this case, the data of the $\mathrm{Diff}(M)$-actions on the spaces of ground states is the data of a truncated TFT $\tau_{<n}\mathsf{Bord}^{\mathrm{O}}_{n,n-1} \to \mathsf{Vect}_{\mathbb{C}}$. We call this *the truncated TFT associated to* this lattice model.

Remark 2.3.2. One must worry about choices here: a priori, the space of ground states of a lattice model on M is only defined up to isomorphism, as it depends on the triangulation in a subtle way. To define a $\mathrm{Diff}(M)$-action or a truncated TFT, we need something more rigid. We show how to do this in §2.3.1.1 below.

In the rest of this subsection, we discuss these $\mathrm{Diff}(M)$-actions. In §2.3.1.1, we recall the definition of the $\mathrm{Diff}(M)$-action on $Z(M)$, and in §2.3.1.2, we address the assumption of the $\mathrm{Diff}(M)$-action on $\ker(H_{M,\Pi})$, showing how to construct such an action given certain data present in the toric code and GDS models. Let $\mathrm{Diff}_0(M) \subset \mathrm{Diff}(M)$ denote the connected component containing the identity; then for both the toric code and GDS models, these $\mathrm{Diff}(M)$-actions factor through $\mathrm{Diff}_0(M)$ and define actions of the *mapping class group* $\mathrm{MCG}(M) \coloneqq \mathrm{Diff}(M)/\mathrm{Diff}_0(M)$.

Remark 2.3.3. The truncated TFT associated to a lattice model is believed to correspond to the physics notion of the low-energy effective theory of the model. The existence of such a low-energy TFT for certain lattice models, called topological phases, is predicted by physics,[9] and the low-energy TFT is expected to determine the lattice model up to some physically meaningful notion of equivalence; this correspondence is discussed in [**FH16a**, **Gai17**, **RW18**, **FT18**].

However, there is much left to understand, especially at a mathematical level of rigor. We do not intend for Definition 2.3.1 to be a mathematical definition of the physical notion of the low-energy effective theory of a lattice model. Providing such a mathematical definition is a major open question; as is, Definition 2.3.1 fails to address uniqueness (as shown in Remark 2.3.61) and existence (due to fracton phases; see, e.g., [**BLT11**, **Haa11**, **Yos13**]).

2.3.1.1. *The mapping class group action for TFTs.* For any $\varphi \in \mathrm{Diff}(M)$, let C_φ denote the *mapping cylinder* of φ, i.e. the bordism $[0,1] \times M$ from M to itself, where M is attached via the identity at 0 and via φ at 1.

If $Z\colon \mathsf{Bord}_{n,n-1} \to \mathsf{Vect}_{\mathbb{C}}$ is a TFT, then the assignment $\varphi \mapsto Z(C_\varphi)\colon Z(M) \to Z(M)$ defines an action of $\mathrm{Diff}(M)$ on $Z(M)$. If $\varphi \in \mathrm{Diff}_0(M)$, then there is a smooth isotopy $\varphi_t\colon [0,1] \times M \to M$ such that $\varphi_t(0,x) = x$

[9]One should allow TFTs tensored with an invertible, non-topological theory, as in [**FH16a**, §5.4]. The truncated TFTs we find in this paper are topological, so this distinction will not matter here.

and $\varphi_t(1,x) = \varphi(x)$, and in particular there is a diffeomorphism of bordisms $C_{\mathrm{id}} \cong C_\varphi$ defined by the map

$$\begin{aligned} [0,1] \times M &\to [0,1] \times M \\ (t,x) &\mapsto (t, \varphi_t(x)). \end{aligned} \tag{2.3.4}$$

Therefore $Z(C_\varphi) = Z(C_{\mathrm{id}}) = \mathrm{id}$, so this $\mathrm{Diff}(M)$-action is trivial on $\mathrm{Diff}_0(M)$, hence defines an $\mathrm{MCG}(M)$-action on $Z(M)$.

2.3.1.2. *The* $\mathrm{Diff}(M)$*-action for a lattice model.* We will imitate the first half of the above argument for a lattice model with some assumptions, constructing a $\mathrm{Diff}(M)$-action on the space of ground states of the model on M; in §§2.3.2.2 and 2.3.3.3, we will see that for the toric code and GDS models, these are trivial when restricted to $\mathrm{Diff}_0(M)$, defining actions of the mapping class group on the spaces of ground states of the toric code and GDS models.

We require the following of our lattice model.

- (A1) The model is defined for closed $(n-1)$-manifolds equipped with a lattice, which here means a CW structure or a triangulation, or one of these structures subject to some condition that can be satisfied on all closed $(n-1)$-manifolds and for which any two such structures on a manifold admit a common refinement.
- (A2) Given a closed manifold M, a diffeomorphism $f\colon M \to M$ and a lattice Π on M, let $f(\Pi)$ denote the lattice obtained by postcomposing the attaching maps in Π with f. We ask for f to induce an isomorphism f_* from the state space of the model for Π to the state space of the model for $f(\Pi)$, for f_* to intertwine the Hamiltonians of these models, and for this to be functorial under composition of diffeomorphisms.
- (A3) Data of, for every refinement $\Pi \to \Pi'$ of lattices, an isomorphism from the space of low-energy states of the model on Π to the space of low-energy states of the model on Π', which is functorial under composition of refinements, and which is compatible with the maps f_* in (A2).

Examples of conditions satisfying the constraint in (A1) include regular CW complexes and the class of smooth triangulations we considered when defining the GDS model.

With these assumptions in place, we define a category $\mathsf{Lat}(M)$ whose objects are the lattices on a closed manifold M and whose morphisms are generated by refinements and diffeomorphisms. Specifically, we add a morphism $r_{\Pi,\Pi'}\colon \Pi \to \Pi'$ for each refinement $\Pi \to \Pi'$, and for each diffeomorphism $f\colon M \to M$ we add a morphism $f_*\colon \Pi \to f(\Pi)$. These morphisms are subject to the relations establishing functoriality under composition of diffeomorphisms and under composition of refinements, and that $f_* \circ r_{\Pi,\Pi'} = r_{f(\Pi),f(\Pi')} \circ f_*$.

Then (A2) and (A3) define a functor $L\colon \mathsf{Lat}(M) \to \mathsf{Vect}_{\mathbb{C}}$ sending a lattice Π to the space of low-energy states of the model on Π; let $Z(M) := \varinjlim L$. Let $f \in \mathrm{Diff}(M)$. If $r_{\Pi,\Pi'}\colon \Pi \to \Pi'$ is a refinement, the fact that $f_* \circ r_{\Pi,\Pi'} = r_{f(\Pi),f(\Pi')} \circ f_*$ means that the action of f_* passes to the colimit, defining a map $f_*\colon Z(M) \to Z(M)$, and this is functorial with respect to diffeomorphisms, defining a $\mathrm{Diff}(M)$-action on $Z(M)$.

2.3.2. Review for the toric code. As a warmup, before tackling the GDS model, we determine a TFT which captures the ground states of the toric code. Neither the answer nor this perspective on it are new.

Theorem 2.3.5. *The spaces of ground states of the toric code assemble into a truncated TFT L^{TC}. If $\mathrm{DW}_0\colon \mathsf{Bord}_{n,n-1} \to \mathsf{Vect}_{\mathbb{C}}$ denotes the $\mathbb{Z}/2$-Dijkgraaf-Witten theory with Lagrangian equal to 0, then $\tau\mathrm{DW}_0 \simeq L^{\mathrm{TC}}$.*

Recall that an equivalence of truncated TFTs means an isomorphism of state spaces with diffeomorphism group representations, and that the state space of L^{TC} is the space of ground states of the toric code on M.

Remark 2.3.6. This is not a new result. Because researchers consider different formulations of the toric code, there are some analogues of Theorem 2.3.5 in the literature for different classes of toric code models, e.g. in [**Kit03, BK12, Cha14**]. Though these results do not cover Theorem 2.3.5 in the case $n > 3$ or mapping class group actions when $n = 3$, Theorem 2.3.5 and its proof were certainly known before this paper. Also, Bartlett-Goosen [**BG20**, Corollary 29] compute the mapping class group actions in a different way when $n = 3$.

Our proof of Theorem 2.3.5 will be slightly more complicated than necessary. This is so that it follows the same line of argument as the proof for the GDS model in §2.3.3. We hope that presenting the simpler example first makes the GDS example easier to understand.

Before we prove Theorem 2.3.5, we must define the $\mathrm{Diff}(M)$-action on $L^{\mathrm{TC}}(M)$. First, though, in §2.3.2.1, we show $L^{\mathrm{TC}}(M) \cong \mathrm{DW}_0(M)$ as vector spaces. Then, in §2.3.2.2, we use the argument of §2.3.1.2 to produce a $\mathrm{Diff}(M)$-action on $L^{\mathrm{TC}}(M)$, compare it with the $\mathrm{MCG}(M)$-action on $\mathrm{DW}_0(M)$, and conclude.

2.3.2.1. *Identifying the vector spaces for the toric code.* Though we have not defined the truncated TFT L^{TC} yet, we will abuse notation slightly in this subsection, letting $L^{\mathrm{TC}}(M)$ denote the space of ground states of the toric code on a manifold M with CW structure. We will see in the proof of Proposition 2.3.7 that as a vector space, $L^{\mathrm{TC}}(M)$ does not depend on the CW structure.

Our goal is to prove the following proposition.

Proposition 2.3.7. *For a closed manifold M, there is an isomorphism of vector spaces $L^{\mathrm{TC}}(M) \cong \mathrm{DW}_0(M)$.*

We can use the fact that the vertex and face operators commute to simplify our analysis of the Hamiltonian.

Lemma 2.3.8. *Let V be a vector space over a field k, and let $\Phi = \sum_{i=1}^m \phi_i$ be a finite sum of commuting projections $\phi_i \in \mathrm{End}_k(V)$. Then, $\ker(\Phi) = \bigcap_{i=1}^m \ker(\phi_i)$.*

Proof. By induction, it suffices to consider $m = 2$, so $\Phi = \phi_1 + \phi_2$. Clearly $\ker(\phi_1) \cap \ker(\phi_2) \subset \ker(\Phi)$, so assume $\Phi x = 0$ for some $x \in V$. Thus $\phi_1 x = -\phi_2 x$, so $\phi_1 x = \phi_1^2 x = -\phi_1\phi_2 x = -\phi_2(\phi_1 x)$, so $\phi_1 x$ is an eigenvector for ϕ_2 with eigenvalue -1. This means $\phi_2^2(\phi_1 x) = (-1)^2\phi_1 x = \phi_1 x$, and since ϕ_2 is a projection, $\phi_2^2(\phi_1 x) = \phi_2\phi_1 x = -\phi_1 x$, forcing $\phi_1 x = 0$. Since $\phi_2 = \Phi - \phi_1$, then $\phi_2 x = 0$ as well. $\square$

Proof of Proposition 2.3.7. Let M be a closed manifold with a CW structure Ξ. As before, we will write (P, ξ) for an object of $\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$, meaning that $P \to M^1$ is a principal $\mathbb{Z}/2$-bundle and $\xi\colon M^0 \to P|_{M^0}$ is a trivialization of P over M^0.

By Lemma 2.3.8, $L^{\mathrm{TC}}(M)$ is isomorphic to the space of functions ψ on $\mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$ such that $H_v\psi = 0$ for all 0-cells v and $H_f\psi = 0$ for all 2-cells f.

Let f be a 2-cell. Then, $H_f\psi = 0$ if and only if $B_f\psi = \psi$, or for all $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$, $(-1)^{\mathrm{Hol}_P(f)}\psi(P, \xi) = \psi(P, \xi)$. That is, either $\psi(P, \xi) = 0$ or $\mathrm{Hol}_P(f) = 0$, so ψ must vanish on all principal $\mathbb{Z}/2$-bundles with nontrivial holonomy around ∂f. Hence if $\psi \in \ker(H_f)$ for all 2-cells f, it can only be nonzero on the principal $\mathbb{Z}/2$-bundles with no holonomy around the boundary of any 2-cell, which are exactly the principal $\mathbb{Z}/2$-bundles which extend to M^2, hence to all of M, and such an extension is necessarily unique. That is, $\bigcap_f \ker(H_f)$ is the space of functions on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$.

Let $\mathcal{A} \coloneqq C^0_\Xi(M; \mathbb{Z}/2)$ denote the group of cellular 0-cochains. We will describe the ground states of the toric code for M as invariant sections of an $\mathcal{A}$-equivariant line bundle on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$, then take the quotient by $\mathcal{A}$. For $v \in \Delta^0(M)$, let $\delta_v \in \mathcal{A}$ be the function equal to 1 on v and 0 elsewhere. Then, $\mathcal{A}$ has a presentation by the following generators and relations:

$$\mathcal{A} \cong \langle \delta_v \text{ for all } v \in \Delta^0(M) \mid \delta_v^2, [\delta_v, \delta_w] \rangle, \tag{2.3.9}$$

so an $\mathcal{A}$-action is the same data as commuting involutions associated to each δ_v. For example, $\mathcal{A}$ acts on the (discrete) groupoid $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ through the commuting involutions

$$\delta_v\colon (P, \xi) \mapsto (P, (w \mapsto \xi(w) + \delta_v(w))). \tag{2.3.10}$$

Consider the trivial line bundle $\underline{\mathbb{C}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ and give it the trivial $\mathcal{A}$-action. We can identify sections of $\underline{\mathbb{C}}$ with functions on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$, and the $\mathcal{A}$-actions match; in particular, if $\psi \in \Gamma(\underline{\mathbb{C}})$ and v is a 0-cell, then $\delta_v \cdot \psi = A_v\psi$. Therefore ψ is invariant under the $\mathcal{A}$-action if and only if $A_v\psi = \psi$ for all v, i.e. $H_v\psi = 0$ for all v. That is, the space of ground states is the space of $\mathcal{A}$-invariant sections of $\underline{\mathbb{C}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$.

The $\mathcal{A}$-equivariant line bundle $\underline{\mathbb{C}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ descends to a nonequivariant line bundle on the groupoid quotient $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)/\mathcal{A}$; since we began with the trivial $\mathcal{A}$-action, this will also be a trivial line bundle. Therefore it suffices to identify the quotient.

Lemma 2.3.11. *The map* $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)/\mathcal{A} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$ *which forgets the trivialization is an equivalence of groupoids. Given* $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ *and* $\phi \in \mathrm{Aut}(P)$*, action by*

$$t_\phi := \sum_{\substack{v\in\Delta^0(M)\\ \phi|_v \text{ nontrivial}}} \delta_v \in \mathcal{A} \tag{2.3.12}$$

on (P, ξ) *passes to* ϕ *in the quotient.*

Proof. $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ is a discrete groupoid, so we just have to determine the stabilizer subgroup for the $\mathcal{A}$-action. An automorphism ϕ of P switches the trivializations wherever ϕ is nontrivial, so defines an isomorphism $(P, \xi) \xrightarrow{\cong} (P, t_\phi \cdot \xi)$. To check these are the only isomorphisms that occur, suppose $(P, \xi) \cong (P, t \cdot \xi)$ for some $t \in \mathcal{A}$. Since the function $\mathrm{spin}_{(P,\xi)}$ is an isomorphism invariant of $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$, t must be the sum of δ_v as v ranges over a set S of 0-cells such that every 1-cell of M bounds an even number of 0-cells in S. Thus for any connected component M_0 of M, S includes either all 0-cells of M_0 or none, so t is realized by some t_ϕ. ☐

Therefore $L^{\mathrm{TC}}(M)$ is isomorphic to the space of sections of $\underline{\mathbb{C}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$, i.e. the space of functions on $\mathrm{Bun}_{\mathbb{Z}/2}(M)$, which is what DW_0 assigns to M. ☐

2.3.2.2. *The* $\mathrm{MCG}(M)$*-action for the toric code.* Recall the axioms (A1)–(A3) from §2.3.1.2 that allow us to produce a $\mathrm{Diff}(M)$-action on $L^{\mathrm{TC}}(M)$. It is clear how to satisfy (A1) and (A2); turning to (A3), a refinement $\varphi\colon \Xi \to \Xi'$ of CW structures on M induces a pullback map

$$\varphi^*\colon\ \mathrm{Bun}_{\mathbb{Z}/2}(M^1_{\Xi'}, M^0_{\Xi'}) \to \mathrm{Bun}_{\mathbb{Z}/2}(M^1_{\Xi}, M^0_{\Xi}). \tag{2.3.13}$$

hence a pushforward map on state spaces: $\varphi_*\colon \mathcal{H}(\Xi) \to \mathcal{H}(\Xi')$.

Remark 2.3.14. The pushforward φ_* does not restrict to an isomorphism on the spaces of ground states. Consider the refinement $\Xi \to \Xi'$ in Figure 6 and (P, ξ) which induce the indicated spins on the 1-cells of Ξ'.

If f is a ground state for Ξ', it must vanish on (P, ξ), because (P, ξ) has nontrivial holonomy around the boundaries of the pictured 2-cells, but pulled back to Ξ, this is no longer the case. Therefore $\mathrm{Im}(\varphi_*)$ contains states which do not vanish on (P, ξ), hence are not ground states.

The issue is that functions in the image of φ_* may not vanish on bundles with nontrivial holonomy around certain boundaries of 2-cells, so in order to satisfy (A3), we zero out their values on any such bundle. Let $\mathcal{P}\colon \mathcal{H}_{\Xi'} \to \mathcal{H}_{\Xi'}$ denote this projection: that is, if $f \in \mathcal{H}_{\Xi'}$ and $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1_{\Xi'}, M^0_{\Xi'})$, let

$$(\mathcal{P}f)(P,\xi) \coloneqq \begin{cases} f(P,\xi), & \text{if } \mathrm{Hol}_P(e) = 0 \text{ for all } e \in \Delta^2(M;\Xi'), \\ 0, & \text{otherwise.} \end{cases} \tag{2.3.15}$$

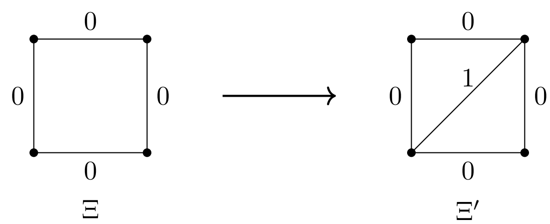


Figure 6. Consider a refinement $\Xi \to \Xi'$ of CW structures as above, together with $(P, \xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M^1_{\Xi'}, M^0_{\Xi'})$ such that the labels on the 1-simplices represent $\mathrm{spin}_{(P,\xi)}$, as in Remark 2.1.25. In Remark 2.3.14, we discuss how (P, ξ) illustrates a subtlety in defining the map from the ground states of the toric code for Ξ to those on Ξ'.

Lemma 2.3.16. *The map $\mathcal{P} \circ \varphi_*$ sends ground states to ground states, hence restricts to an isomorphism $L(\Xi) \xrightarrow{\cong} L(\Xi')$ functorial in the sense of* (A3)*, and this is compatible with the maps in* (A2)*.*

Proof. Let $f \in L(\Xi)$. By construction $\mathcal{P}(\varphi_*(f))$ vanishes on principal $\mathbb{Z}/2$-bundles with nontrivial holonomy, so it suffices to check that it does not depend on the trivializations on the 0-cells. This is not changed by $\mathcal{P}$, so we can just think about $\varphi_*(f)$. Let $v \in \Delta^0(M, \Xi')$ and suppose v is also a 0-cell of Ξ. Then $\varphi_*(f)$ cannot depend on the trivialization at v, because f does not depend on the trivialization at v. If instead v is not a 0-cell of Ξ, so is created by the refinement, then $\varphi_*(f)$ also does not depend on the trivialization at v, because $\varphi_*(f)(P, \xi)$ is computed by pulling back to Ξ, where v is not a cell. $\square$

Therefore the argument of §2.3.1.2 applies to define for any closed $(n-1)$-manifold M an action of $\mathrm{Diff}(M)$ on $L^{\mathrm{TC}}(M)$. Under the identification of this space with $\mathbb{C}[\mathrm{Bun}_{\mathbb{Z}/2}(M)]$, this representation is the one induced from the usual $\mathrm{Diff}(M)$-action on $\pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M) \cong H^1(M; \mathbb{Z}/2)$, which is trivial on the subgroup $\mathrm{Diff}_0(M)$ and therefore defines an action of the mapping class group.

Recall from Proposition 2.3.7 that for any closed $(n-1)$-manifold M, the state space of $\mathbb{Z}/2$-Dijkgraaf-Witten theory with Lagrangian equal to 0 on M, denoted $\mathrm{DW}_0(M)$, is isomorphic to the space of ground states of the toric code on M. Explicitly, $\mathrm{DW}_0(M) \cong \mathbb{C}[\mathrm{Bun}_{\mathbb{Z}/2}(M)]$, and DW_0 assigns to a bordism a push-pull map, which implies that the $\mathrm{MCG}(M)$-action on $\mathrm{DW}_0(M)$ is also the action induced from the standard action on $\pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$. Therefore the identification $L^{\mathrm{TC}}(M) \cong \mathrm{DW}_0(M)$ in Proposition 2.3.7 is equivariant with respect to the $\mathrm{MCG}(M)$-actions on both sides, proving Theorem 2.3.5.

Remark 2.3.17. The mapping class group action determines the partition functions of mapping tori: if $f \in \mathrm{MCG}(M)$, then $Z(M_f)$ is the trace of f acting on $Z(M)$. Though we can see these partition functions from the lattice, it is not clear in general how to extend this to arbitrary closed n-manifolds.

2.3.3. Derivation of the generalized double semion Lagrangian. We now turn to the main goal of this chapter: extracting a truncated TFT from the GDS model and computing what truncated TFT it is.

Definition 2.3.18. Fix a dimension n. Let $\alpha \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$ denote the generator and $w \in H^*(B\mathrm{O}_n;\mathbb{Z}/2)$ denote the total Stiefel-Whitney class. In $H^*(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$, α is nilpotent, so $1+\alpha$ is invertible, and we can consider $w\alpha/(1+\alpha) \in H^*(B\mathrm{O}_n \times B\mathbb{Z}/2;\mathbb{Z}/2)$, which is a sum of homogeneous elements of different degrees. Let β denote the degree-n summand of $w\alpha/(1+\alpha)$. We let $Z_{\mathrm{GDS}}\colon \mathsf{Bord}^{\mathrm{O}}_{n,n-1} \to \mathsf{Vect}_{\mathbb{C}}$ denote the quantum gauge-gravity theory Z_β from Definition 2.2.22; the dimension n will be clear from context when needed.

Our goal in this section is to prove the following.

Theorem 2.3.19. *The spaces of ground states of the GDS model assemble into a truncated TFT L^{GDS}, and there is an equivalence of truncated TFTs $\tau Z_{\mathrm{GDS}} \cong L^{\mathrm{GDS}}$.*

As with the toric code, we first establish an isomorphism of vector spaces in §§2.3.3.1 and 2.3.3.2. Then, in §2.3.3.3, we invoke the argument of §2.3.1.2 to define the $\mathrm{Diff}(M)$-action on the space of ground states of the GDS model on a closed manifold M and compare it with the action on Z_{GDS}, finishing the proof of Theorem 2.3.19. We will again abuse notation and, given a manifold M with smooth triangulation, we will write $L^{\mathrm{GDS}}(M)$ for the space of ground states of the GDS model on M; in the course of Theorem 2.3.20, we will see that up to isomorphism, this does not depend on the triangulation.

2.3.3.1. *Defining $L_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$.* Our first goal is to prove the following theorem.

Theorem 2.3.20. *For a closed manifold M, there is an isomorphism of vector spaces $L^{\mathrm{GDS}}(M) \cong Z_{\mathrm{GDS}}(M)$.*

Let M be a closed $(n-1)$-manifold with a smooth triangulation Π; as in §2.1.1, we assume the 0-clopen star of any vertex is contractible. We will prove Theorem 2.3.20 by identifying $L^{\mathrm{GDS}}(M)$ with the space of sections of a line bundle $L_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$ defined below. Proposition 2.2.23 identifies $Z_{\mathrm{GDS}}(M)$ with the sections of another line bundle $L_\beta \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$, and we will show that $L_{\mathrm{GDS}} \cong L_\beta$.

The commutativity relations for the operators in the GDS model are more complicated than those for the toric code, but we can still understand the spaces of ground states in terms of the vertex and face operators.

Lemma 2.3.21. *With V as in Lemma 2.3.8, let $\phi_i, \psi_j \in \mathrm{End}_k(V)$ and suppose*

$$H = \underbrace{\sum_{i=1}^{\ell} \phi_i}_{\Phi} + \underbrace{\sum_{j=1}^{m} \psi_j}_{\Psi}, \tag{2.3.22}$$

such that for all i and j,

(1) ϕ_i and ψ_j are projections,

(2) $[\phi_i, \phi_j] = 0$,

(3) $[\phi_i, \psi_j] = 0$,

(4) for any $x \in \ker(\Phi)$, $[\psi_i, \psi_j]x = 0$.

Then,

$$\ker(H) = \bigcap_{j=1}^{m} \ker(\psi_j \colon \ker(\Phi) \to \ker(\Phi)). \tag{2.3.23}$$

Proof. Lemma 2.3.8 tells us $\ker(H) = \ker(\Phi) \cap \ker(\Psi)$, so it suffices to restrict to $\ker(\Phi)$. Since ϕ_i and ψ_j commute, then $\psi_j(\ker\Phi) \subset \ker\Phi$ for each j, so we may consider ψ_j as an operator on $\ker(\Phi)$. Restricted to this subspace, $[\psi_i, \psi_j] = 0$, so we apply Lemma 2.3.8 again to conclude. ☐

The upshot is that for a Hamiltonian whose smallest eigenvalue is 0 and which is a sum of vertex and face operators satisfying the commutativity conditions in Lemma 2.3.21, the space of ground states can be computed by finding the $f \in \mathcal{H}$ with $\phi_i f = 0$ for all i, then taking the subspace of those such that $\psi_j f = 0$ for all j. By Lemmas 2.1.20 and 2.1.26, the vertex and face operators for the GDS model satisfy the commutation relations in Lemma 2.3.21, where the ϕ_i are the face operators and the ψ_j are the vertex operators, so we will use this method to find the space of ground states.

The first part of the derivation is to determine $\bigcap_f \ker(H_f)$. The H_f operators in the GDS model are the same as in the toric code, so the derivation proceeds as for the toric code (the first part of the proof of Theorem 2.3.5) to produce the space of functions on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$.

Next, we will use the vertex operators to define $L_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$ and characterize the ground states on M as its space of sections. Specifically, letting $\mathcal{A} \coloneqq C^0_\Pi(M;\mathbb{Z}/2)$ as in the previous section, we will describe an $\mathcal{A}$-equivariant line bundle on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ whose invariant sections are the ground states, then let $L_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$ denote the induced bundle on the quotient.

Definition 2.3.24. First, we define the $\mathcal{A}$-equivariant line bundle $L'_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$. Begin with the trivial (nonequivariant) line bundle $\underline{\mathbb{C}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$, and give it an $\mathcal{A}$-action as follows: if $(P,\xi) \in \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ and $z \in \mathbb{C}$, let

$$\delta_v \colon ((P,\xi), z) \mapsto (\delta_v \cdot (P,\xi), \sigma(v,(P,\xi))z), \tag{2.3.25}$$

where $\sigma(v,(P,\xi))$ is the GDS sign from (2.1.12). By Lemmas 2.1.20 and 2.1.26, the actions of δ_{v_1} and δ_{v_2} on $\underline{\mathbb{C}}$ commute for 0-cells v_1 and v_2, so (2.3.25) defines an $\mathcal{A}$-action covering the $\mathcal{A}$-action on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$.

Identifying functions on $\mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ with sections of the trivial line bundle, hence of $L'_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$, a section ψ is invariant under the $\mathcal{A}$-action if and only if $\psi \in \ker(\widetilde{H}_v)$ for all $v \in \Delta^0(M)$; hence, by Lemma 2.3.21, this identifies $L^{\mathrm{GDS}}(M)$ with the space $\Gamma(L'_{\mathrm{GDS}})^{\mathcal{A}}$ of invariant sections of L'_{GDS}. By Lemma 2.3.11, $L'_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M, M^0)$ descends to a (nonequivariant) line bundle $L_{\mathrm{GDS}} \to \mathrm{Bun}_{\mathbb{Z}/2}(M)$, and there is an isomorphism $\Gamma(L'_{\mathrm{GDS}})^{\mathcal{A}} \cong \Gamma(L_{\mathrm{GDS}})$, so $L^{\mathrm{GDS}}(M) \cong \Gamma(L_{\mathrm{GDS}})$.

2.3.3.2. *Computing the isomorphism type of* L_{GDS}. Given a principal $\mathbb{Z}/2$-bundle $P \to M$, the action of $\mathrm{Aut}(P)$ on $(L_{\mathrm{GDS}})_P$ is a character of $\mathrm{Aut}(P)$, and the data of these characters for all $P \in \pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$ determines L_{GDS} up to isomorphism. In this section, we compute these characters, describing the answer in Corollary 2.3.58.

Let $P \to M$ be a principal $\mathbb{Z}/2$-bundle and $\phi \in \mathrm{Aut}(P)$. Let $\mathcal{V}$ denote the set of vertices on which ϕ is nontrivial, and order this set as $\{v_1, \dots, v_m\}$. Fix a trivialization ξ_0 of $P|_{M^0}$ and let

$$\xi_i \coloneqq \delta_{v_i} \cdot (\delta_{v_{i-1}} \cdot (\cdots (\delta_{v_1} \cdot \xi_0) \cdots)). \tag{2.3.26}$$

In Lemma 2.3.11, we identified the action of ϕ on L_{GDS} with the action of t_ϕ on L'_{GDS}, which is multiplication by

$$\sigma_{\mathcal{V}} \coloneqq \prod_{i=1}^{m} \sigma(v_i, (P, \xi_i)). \tag{2.3.27}$$

To compare L_{GDS} and L_β, we need to pass from this description of $\sigma_{\mathcal{V}}$ in terms of simplices to a description only depending on M and P. The following theorem makes this transition; afterwards we use characteristic classes to finish the calculation.

As in Proposition 2.2.23, let $P_\phi \to S^1 \times M$ denote the mapping torus of ϕ.

Theorem 2.3.28. *Let $N \subset S^1 \times M$ be an embedded submanifold representing the Poincaré dual to $\alpha(P_\phi) \in H^1(S^1 \times M; \mathbb{Z}/2)$. Then $\sigma_{\mathcal{V}} = (-1)^{\chi(N)}$.*

Our proof has two parts.

(1) First, the simplicial part: we construct an $(n-1)$-cycle C on $S^1 \times M$, cellular with respect to a certain CW structure, which represents the Poincaré dual of $\alpha(P_\phi)$ (Lemma 2.3.34) and such that if $|C|$ denotes the geometric realization of C, then $\sigma_{\mathcal{V}} = (-1)^{\chi(|C|)}$ (Proposition 2.3.37).

(2) Then, we show that replacing $|C|$ with a smoothly embedded representative of the homology class of C does not change the mod 2 Euler characteristic (Proposition 2.3.47).

The proof employs the dual CW structure $\Pi^\vee$ to the given triangulation Π; see Remark 2.1.15 for more information. Let $S^1(m)$ denote the simplicial structure on S^1 with m vertices, and choose an identification of the vertices with $\mathbb{Z}/m$ such that i and $i+1$ mod m share an edge for each i. Then let $S^1(m) \times \Pi^\vee$ denote the product CW structure.

For any $i \in \mathbb{Z}/m$, the cellular 1-cochain $\mathrm{spin}_{(P,\xi_i)} \colon \Delta^1(M;\Pi) \to \mathbb{Z}/2$ is a cocycle representative for $\alpha(P) \in H^1(M;\mathbb{Z}/2)$, and therefore

$$Y_i \coloneqq \{e^\vee \mid e \in \Delta^1(M;\Pi) \text{ and } \mathrm{spin}_{(P,\xi_i)}(e) = 1\} \subset \Delta^{n-2}(M;\Pi^\vee) \tag{2.3.29}$$

is a cellular $(n-2)$-cycle representative for the Poincaré dual of $\alpha(P)$ in $H_{n-2}(M;\mathbb{Z}/2)$. From the definitions of Y_i and of ξ_i (2.3.26) we see that

$$Y_i = Y_{i-1} + \partial v_i^\vee, \tag{2.3.30}$$

where $i-1$ is interpreted in $\mathbb{Z}/m$, and that

$$C \coloneqq \sum_{i \in \mathbb{Z}/m} \left((i,i+1) \times Y_i + \{i\} \times v_i^\vee\right) \subset \Delta^n(S^1 \times M; S^1(m) \times \Pi^\vee) \tag{2.3.31}$$

is a cellular $(n-1)$-cycle on $S^1 \times M$.

Definition 2.3.32. If $P \to M$ is a principal $\mathbb{Z}/2$-bundle over a closed manifold M, there is an isomorphism $\mathrm{Aut}(P) \to H^0(M;\mathbb{Z}/2)$ sending $\phi \in \mathrm{Aut}(P)$ to the function on $\pi_0(M)$ which is 0 on a connected component if ϕ is trivial there and 1 if ϕ is nontrivial there. The image of $\phi \in \mathrm{Aut}(P)$ under this isomorphism is denoted $[\phi]$.

For example, if $x \in H^1(S^1; \mathbb{Z}/2)$ denotes the generator, then

$$\alpha(P_\phi) = \alpha(P) + x[\phi] \in H^1(S^1 \times M; \mathbb{Z}/2). \tag{2.3.33}$$

Lemma 2.3.34. *The homology class C represents is the Poincaré dual of $\alpha(P_\phi) \in H^1(S^1 \times M; \mathbb{Z}/2)$.*

Proof. Recall that $Y_0 \subset \Delta^{n-2}(M; \Pi^\vee)$ is a cellular $(n-2)$-cycle representing the Poincaré dual of $\alpha(P) \in H^1(M; \mathbb{Z}/2)$. The $(n-1)$-cycle in $S^1 \times M$ defined to be the set of $(n-1)$-cells of

$$(S^1 \times |Y_0|) \cup \bigcup_{\substack{M_i \in \pi_0(M) \\ [\phi](M_i)=1}} \{0\} \times M_i \tag{2.3.35}$$

represents the Poincaré dual to $\alpha(P) + x[\phi] = \alpha(P_\phi)$ (2.3.33), and is homologous to C in $Z_{n-1}^{S^1(m)\times\Pi^\vee}(S^1 \times M; \mathbb{Z}/2)$ by adding boundaries of the form $\partial((0,i) \times v_i^\vee)$. □

Lemma 2.3.36. *For $1 \le i \le m$, let $Z_{v_i}(P, \xi_i)$ be as in Proposition 2.1.28. Then $\#(\overline{Y_i} \cap \partial v_i^\vee) = \#(Z_{v_i}(P, \xi_i))$ and therefore $(-1)^{1+\chi(|Y_i| \cap \partial v_i^\vee)} = \sigma(v_i, (P, \xi_i))$.*

Proof. This is a matter of unwinding the definitions: $c \in \overline{Y_i} \cap \partial v_i^\vee$ means that $v_i \in \partial c^\vee$ and either

1. c is an $(n-2)$-cell and $\mathrm{spin}_{(P,\xi_i)}(c^\vee) = 1$, or
2. there is an $(n-2)$-cell $e \in Y_i$ with $c \in \partial e$, i.e. $\mathrm{spin}_{(P,\xi_i)}(e^\vee) = 1$ and $e^\vee \in \partial c^\vee$.

These are exactly the conditions for $c^\vee$ to be in $Z_{v_i}(P, \xi_i)$, so $\#(\overline{Y_i} \cap \partial v_i) = \#(Z_{v_i}(P, \xi_i))$, and the rest of the conclusion then follows from Proposition 2.1.28. □

Proposition 2.3.37. $(-1)^{\chi(|C|)} = \sigma_{\mathcal{V}}$.

Proof. The projection map $\pi\colon S^1 \times M \twoheadrightarrow S^1$ is cellular with respect to $S^1(m) \times \Pi^\vee$ and $S^1(m)$; if $D_i := |C| \cap \pi^{-1}([i, i+1))$, then each D_i is a union of cells and

$$|C| = \coprod_{i \in \mathbb{Z}/m} D_i. \tag{2.3.38}$$

Define A_i and B_i by $\pi^{-1}(\{i\}) = \{i\} \times A_i$ and $\pi^{-1}((i, i+1)) = (i, i+1) \times B_i$; A_i and B_i are also unions of cells. Then

$$A_i = |Y_i| \cup |Y_{i-1}| \cup |v_i^\vee| = |Y_i| \cup |v_i^\vee| \tag{2.3.39a}$$

because $Y_{i-1} = Y_i + \partial v_i^\vee$ (2.3.30), and

$$B_i = |Y_i|. \tag{2.3.39b}$$

Therefore

$$\begin{aligned}
\#(\text{cells of } D_i) &= \#(\text{cells of } A_i) + \#(\text{cells of } B_i)\\
&= \chi(|Y_i| \cup |v_i^\vee|) + \chi(|Y_i|)\\
&= \chi(|Y_i| \cup \operatorname{int}(|v_i^\vee|) \cup |\partial v_i^\vee|) + \chi(|Y_i|)\\
&= 1 + \chi(|Y_i|) + \chi(|\partial v_i^\vee|) - \chi(|Y_i| \cap |\partial v_i^\vee|) + \chi(|Y_i|)\\
&\equiv_2 1 + \chi(|Y_i| \cap |\partial v_i^\vee|),
\end{aligned} \tag{2.3.40}$$

since $\partial v_i^\vee \cong S^{n-1}$, which has even Euler characteristic. Looking at the definition of $\sigma_{\mathcal{V}}$ from (2.3.27), it suffices to equate $(-1)^{1+\chi(|Y_i| \cap \partial v_i^\vee)}$ with $\sigma(v_i, (P, \xi_i))$, which is taken care of by Lemma 2.3.36. $\square$

Now we show that we can replace $|C|$ with a smooth representative of the homology class of C.

Definition 2.3.41. Let M be a smooth manifold and $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. A C^r *triangulation* of M is a triangulation $(K, f\colon |K| \to M)$ of M such that for every simplex e of K, $f|_{|e|}$ is a C^r map.

Theorem 2.3.42 (Munkres [Mun66, Theorem 10.6])**.** *Let W be a compact manifold and $r \in \mathbb{Z}_{>0} \cup \{\infty\}$. Then every C^r triangulation of ∂W extends to a C^r triangulation of W.*

Corollary 2.3.43. *Let X be a closed smooth manifold and $Y \subset X$ be a smooth codimension-one submanifold. Then there is a triangulation of X such that Y is a union of simplices.*

Proof. Let $\nu \to Y$ denote the normal bundle of $Y \hookrightarrow X$, $D(\nu) \to Y$ denote the unit disc bundle of ν, and $S(\nu) \to Y = \partial D(\nu)$ denote the unit sphere bundle of ν. Using the tubular neighborhood theorem, we choose an embedding $i\colon D(\nu) \hookrightarrow M$ such that the original embedding of Y in X is the zero section of $D(\nu) \to Y$ followed by i.

Let $r \geq 1$. Given a C^r triangulation $\Pi(N)$ of Y, we can triangulate $D(\nu)$: let $\Pi(I)$ denote the triangulation of $[-1, 1]$ which has vertices precisely at the integers, which is a smooth triangulation. For any simplex e of $\Pi(Y)$, $D(\nu)|_{|e|}$ is isomorphic to $|e| \times [-1, 1]$; choose an isomorphism ψ_e, and give $D(\nu)_{|e|}$ the product triangulation $|e| \times \Pi(I)$. These are compatible as e varies: if e' is another cell and $|e'|$ intersects $|e|$, $(\psi_{e'}^{-1} \circ \psi_e)|_{|e| \cap |e'|}$ is either the identity or multiplication by -1 on the fiber. Both of these send simplices to simplices, so we can glue the triangulations on $D(\nu)|_{|e|}$ and $D(\nu)|_{|e'|}$. Doing this for all simplices of Y defines a C^r triangulation $\Pi(D(\nu))$ of $D(\nu)$ in which $Y \subset D(\nu)$ is a union of simplices.

This induces a C^r triangulation of $S(\nu) = \partial(\overline{X \setminus D(\nu)})$, which by Theorem 2.3.42 extends to a triangulation of $\overline{X \setminus D(\nu)}$. We glue this triangulation to $\Pi(D(\nu))$, since both triangulations agree on $S(\nu)$, to obtain a triangulation of X in which Y is a union of simplices. $\square$

Lemma 2.3.44. *Let Π be a triangulation of an n-manifold X, $C \in Z^{\Pi}_{n-1}(X; \mathbb{Z}/2)$, and $f \in \Delta^n(X)$. Then*

$$\chi(|C|) \equiv \chi(|C + \partial f|) \bmod 2. \tag{2.3.45}$$

Proof. The sets of simplices in $|C|$ and $|C + \partial f|$ agree away from $|f|$, so if $R_0 := |C| \cap |\partial f|$ and $R_1 := |C + \partial f| \cap |\partial f|$, then it suffices to show $\chi(R_0) \equiv \chi(R_1) \bmod 2$.

Inclusion-exclusion implies

$$\chi(R_0) + \chi(R_1) \equiv \chi(|\partial f|) + \chi(R_0 \cap R_1) \bmod 2. \tag{2.3.46}$$

Since $|\partial f| \cong S^{d-1}$, its Euler characteristic is even. Next we show R_0 is a topological manifold with boundary: if R_0 is empty or all of $|\partial f|$, this is clear, and otherwise R_0 is an iterated boundary connect sum of its $(n-1)$-simplices. Since $R_0 \cap R_1 = \partial R_0$, $R_0 \cap R_1$ is null-bordant as a topological manifold, so its Euler characteristic is even, and (2.3.46) simplifies to $\chi(R_0) = \chi(R_1) \bmod 2$. $\square$

Proposition 2.3.47. *With C as in (2.3.31), if $N \hookrightarrow S^1 \times M$ is a smooth representative for the homology class of C (namely, the Poincaré dual of $\alpha(P_\phi)$), then $\chi(|C|) \equiv \chi(N) \bmod 2$.*

Proof. Let Π_1 be the barycentric subdivision of Π; as noted in Remark 2.1.15, this is also a “refinement” of $\Pi^\vee$, in that every cell of $\Pi^\vee$ is a union of simplices of Π_1. By Corollary 2.3.43, there is a triangulation Π_t of M such that N is a union of simplices; let Π' be a common refinement of Π_1 and Π_t, and $S^1(m) \times \Pi'$ be the product triangulation of $S^1 \times M$.

Let $C_{\mathrm{top}} \in Z^{S^1(m)\times\Pi'}_{n-1}(S^1 \times M; \mathbb{Z}/2)$ denote the cycle whose simplices are those contained in the cells of C; then $|C_{\mathrm{top}}| = |C|$. If $C_{\mathrm{sm}} \in Z^{S^1(m)\times\Pi'}_{n-1}(S^1 \times M; \mathbb{Z}/2)$ denotes the $(n-1)$-simplices in N, then $N = |C_{\mathrm{sm}}|$ and C_{top} and C_{sm} are homologous, so there are n-cells $f_1, \dots, f_\ell$ such that

$$C_{\mathrm{sm}} = C_{\mathrm{top}} + \sum_{i=1}^{\ell} \partial f_i. \tag{2.3.48}$$

We apply Lemma 2.3.44 ℓ times and conclude. $\square$

By combining this with Proposition 2.3.37, we have proven Theorem 2.3.28.

Next, we translate $(-1)^{\chi(N)}$ into an expression involving characteristic classes of M and P.

Proposition 2.3.49. *Let M be a closed manifold, $P \to M$ be a principal $\mathbb{Z}/2$-bundle, and $N \subset M$ be a smoothly embedded, codimension-1 submanifold representing the Poincaré dual to $\alpha(P)$. Then,*

$$\chi(N) \bmod 2 = \left\langle \frac{w(M)\alpha(P)}{1+\alpha(P)}, [M] \right\rangle. \tag{2.3.50}$$

But before we prove this:

Lemma 2.3.51. *Let $L \to X$ be a line bundle over a closed manifold X and $Y \hookrightarrow X$ be a smoothly embedded closed submanifold representing the Poincaré dual to $w_1(L)$, with normal bundle $\nu \to Y$. Then, as line bundles over Y, $\nu \cong L|_Y$.*

Proof. If $i_!\colon H^*(Y;\mathbb{Z}/2) \hookrightarrow H^{*+1}(X;\mathbb{Z}/2)$ denotes the Gysin map (which is Poincaré dual to restriction $H^*(X;\mathbb{Z}/2) \to H^*(Y;\mathbb{Z}/2)$), then $i_!(1)$ is Poincaré dual to $[Y] \in H_{d-1}(X;\mathbb{Z}/2)$ and $i^*i_!(1) = w_1(\nu)$. By construction, $[Y]$ is Poincaré dual to $w_1(L)$, so $i^*w_1(L) = w_1(L|_Y) = w_1(\nu)$. As line bundles are classified by their Stiefel-Whitney classes, $\nu \cong L|_Y$. $\square$

Proof of Proposition 2.3.49. Let $j\colon N \hookrightarrow M$ be inclusion. Since N represents the Poincaré dual of $\alpha(P)$, then for any $x \in H^{n-1}(M;\mathbb{Z}/2)$,

$$\langle j^*x, [N]\rangle = \langle \alpha(P)x, [M]\rangle. \tag{2.3.52}$$

We will use this to carry the mod 2 Euler characteristic of N, which is equal to $\langle w(N), [N]\rangle$, to the cohomology of M; in order to do so, we must show $w(N) \in \mathrm{Im}(j^*)$.

If $\nu \to N$ denotes the normal bundle of N, there is a short exact sequence of vector bundles on N

$$0 \longrightarrow TN \longrightarrow j^*TM \longrightarrow \nu \longrightarrow 0, \tag{2.3.53}$$

so $w(j^*TM) = j^*w(M) = w(N)w(\nu)$. Since ν is a line bundle,

$$w(\nu) = 1 + w_1(\nu) = 1 + j^*\alpha(P) = j^*(1+\alpha(P)) \tag{2.3.54}$$

by Lemma 2.3.51. Hence

$$j^*w(M) = w(N)j^*(1+\alpha(P)). \tag{2.3.55}$$

Since $\alpha(P) \in H^*(X;\mathbb{Z}/2)$ is nilpotent, $j^*(1+\alpha(P))$ is invertible, and therefore

$$w(N) = \frac{j^* w(M)}{j^*(1+\alpha(P))} = j^*\left(\frac{w(M)}{1+\alpha(P)}\right). \tag{2.3.56}$$

Thus we can invoke Poincaré duality:

$$\chi(N) \bmod 2 = \langle w(N), [N]\rangle = \left\langle \alpha(P)\cdot \frac{w(M)}{1+\alpha(P)}, [M]\right\rangle. \tag{2.3.57}$$

□

Combining this with Theorem 2.3.28, we get:

Corollary 2.3.58. *If $P \in \mathrm{Bun}_{\mathbb{Z}/2}(M)$, the character of $\mathrm{Aut}(P)$ acting on $(L_{\mathrm{GDS}})_P$ has ϕ act by multiplication by*

$$(-1)^{\langle \alpha(P_\phi) w(S^1\times M)/(1+\alpha(P_\phi)), [S^1\times M]\rangle} \in \{\pm 1\} \subset \mathbb{C}^\times. \tag{2.3.59}$$

Next, we compare this with the character of $\mathrm{Aut}(P)$ acting on $(L_\beta)_P$ and conclude.

Proof of Theorem 2.3.20. Proposition 2.2.23 tells us that in the character of $\mathrm{Aut}(P)$ acting on $(L_\beta)_P$, ϕ acts by $Z_\beta^{\mathrm{cl}}(S^1\times M, P_\phi)$; by Theorem 2.2.1, this is exactly (2.3.59). Hence $L_{\mathrm{GDS}} \cong L_\beta$. □

2.3.3.3. *The* $\mathrm{MCG}(M)$*-action for the GDS model.* Let $\mathsf{Cell}(M)$ denote the poset category whose objects are smooth triangulations on M such that the 0-clopen star of every vertex is contractible, and whose morphisms are generated by diffeomorphisms and refinements similarly to the construction of $\mathsf{Lat}(M)$ in §2.3.1.2. Just as for the toric code, given a diffeomorphism $f\colon M \to M$ and $\Pi \in \mathsf{Cell}(M)$, we obtain a map f_* from the state space for Π to the state space for $f(\Pi)$, and this assignment satisfies (A2).

Let $\varphi\colon \Pi \to \Pi'$ be a refinement. Define φ_* and $\mathcal{P}$ as in the previous section, and let $\mathcal{P}'\colon \mathcal{H}_{\Pi'} \to \mathcal{H}_{\Pi'}$ be the projection onto $\bigcap_v \widetilde{H}_v$ which is orthogonal with respect to the inner product in which the δ-functions on elements of $\pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M^1, M^0)$ are an orthonormal basis.

Lemma 2.3.60. *The map $\mathcal{P}\circ\mathcal{P}'\circ\varphi_*$ sends ground states to ground states, hence restricts to an isomorphism $L(\Pi) \xrightarrow{\cong} L(\Pi')$ functorial as in* (A3)*, and this is compatible with the maps in* (A2)*.*

Proof. Suppose φ adds no 0-simplices and 1-simplices to Π, so $\mathcal{H}_{\Pi'} \cong \mathcal{H}_{\Pi'}$ and φ_* is the identity. Then φ adds no cells at all, because it is not possible to add cells to a manifold that is a simplicial complex without adding 0- or 1-simplices, so φ is the identity refinement and the lemma follows because $\mathcal{P}$ and $\mathcal{P}'$ are projections.

If otherwise, we show that φ_* of a nonzero ground state is not a ground state, so that the orthogonal projection thereafter sends it to a nonzero ground state. If φ adds any 1-simplices to Π that do not arise from splitting preexisting 1-simplices into smaller ones, the construction in Remark 2.3.14 shows that φ_* of a nonzero ground state is not a ground state; if the only 1-simplices it adds are split from preexisting ones, then it must add a 0-simplex. If φ adds any 0-simplices to Π, it must add a 1-simplex that is not split from a preexisting 1-simplex, because all 0-simplices must be trivalent. $\square$

Therefore the argument of §2.3.1.2 applies to define for any closed $(n-1)$-manifold M an action of $\mathrm{Diff}(M)$ on the ground states of the GDS model. Under the identification of the space of ground states with the space of functions on the set of $P \in \pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$ such that $\langle \alpha(P)w(M)/(1+\alpha(P)), [M]\rangle = 0$, this representation is the one induced from the usual $\mathrm{Diff}(M)$-action on this space, which is an invariant subspace of $\mathbb{C}[\mathrm{Bun}_{\mathbb{Z}/2}(M)]$, and once again this is trivial on $\mathrm{Diff}_0(M)$, so it defines an $\mathrm{MCG}(M)$-action.

Recall from Theorem 2.3.20 that $Z_{\mathrm{GDS}}(M)$ is isomorphic to the space of ground states of the GDS model on M; using the push-pull map Z_{GDS} assigns to a bordism, its $\mathrm{MCG}(M)$-action is the same, again induced from the standard action on $\pi_0 \mathrm{Bun}_{\mathbb{Z}/2}(M)$, finishing the proof of Theorem 2.3.19.

Remark 2.3.61. Suppose n is even, and let $Z_2\colon \mathsf{Bord}_{n,n-1}^{\mathrm{O}} \to \mathsf{Vect}_{\mathbb{C}}$ denote the quantum gauge-gravity TFT with Lagrangian β_2 equal to the degree-n summand of $w\alpha/(1+\alpha^2) \in H^*(B\mathrm{O}_n \times B\mathbb{Z}/2)$. Then $Z_{\mathrm{GDS}}(\mathbb{RP}^n) = 1$ and $Z_2(\mathbb{RP}^n) = 0$, so $Z_{\mathrm{GDS}} \neq Z_2$. However, a characteristic-class computation shows that for any closed $(n-1)$-manifold M, there is an isomorphism $Z_{\mathrm{GDS}}(M) \cong Z_2(M)$ equivariant with respect to the $\mathrm{MCG}(M)$-action on the state spaces. This means that in the sense of Definition 2.3.1, both Z_{GDS} and Z_2 capture the ground states of the GDS model, and that it is not clear how to distinguish them using data from the lattice. In physics, however, the low-energy effective theory of a lattice model is expected to be unique.

Freed-Hopkins [**FH16a**, §7.3], following Kong-Wen [**KW14**], suggest that the low-energy effective theory may only be defined on manifolds which locally have a direction of time, i.e. manifolds M together with a reduction of the structure group of TM from O_n to O_{n-1}. That is, it should be possible to calculate the partition function on such manifolds using locality of the lattice model, and it might not be possible to calculate further in general. Alternatively, Shapourian-Shiozaki-Ryu [**SSR17a**] describe a method to compute partition functions on $\mathbb{RP}^2$ for 2D symmetry-protected topological phases defined by a Hamiltonian, and it is possible their method would generalize, though we have not pursued this.

2.4. Calculations

In this section, we perform some calculations with the GDS Lagrangian in order to understand when Z_{GDS} is isomorphic to a $\mathbb{Z}/2$-Dijkgraaf-Witten theory. First, we fix some notation.

- Recall that α denotes the generator of $H^1(B\mathbb{Z}/2;\mathbb{Z}/2)\cong\mathbb{Z}/2$; in particular, it defines a characteristic class for principal $\mathbb{Z}/2$-bundles by pullback, and if $P\in\mathrm{Bun}_{\mathbb{Z}/2}(X)$, this characteristic class evaluated on P is denoted $\alpha(P)\in H^1(X;\mathbb{Z}/2)$.
- $\mathrm{DW}_0\colon \mathsf{Bord}_{n,n-1}^{\mathrm{O}}\to\mathsf{Vect}_{\mathbb{C}}$ denotes $\mathbb{Z}/2$-Dijkgraaf-Witten theory with the zero Lagrangian and $Z_{\alpha^n}\colon \mathsf{Bord}_{n,n-1}^{\mathrm{O}}\to\mathsf{Vect}_{\mathbb{C}}$ denotes $\mathbb{Z}/2$-Dijkgraaf-Witten theory with Lagrangian $\alpha^n\in H^n(B\mathbb{Z}/2;\mathbb{Z}/2)$.
- Recall from Definition 2.3.32 that if $P\to M$ is a principal $\mathbb{Z}/2$-bundle, the image of $\phi\in\mathrm{Aut}(P)$ under the isomorphism $\mathrm{Aut}(P)\to H^0(M;\mathbb{Z}/2)$ is denoted $[\phi]$. Letting $x\in H^1(S^1;\mathbb{Z}/2)$ denote the generator, $\alpha(P_\phi)=\alpha(P)+x[\phi]$ in $H^*(S^1\times M;\mathbb{Z}/2)$.

We begin with a few example calculations. We will call a principal $\mathbb{Z}/2$-bundle $P\to M$ *permitted* if the GDS action $\langle w(M)\alpha(P_\phi)/(1+\alpha(P_\phi)),[M]\rangle$ vanishes for all $\phi\in\mathrm{Aut}(P)$; thus $Z_{\mathrm{GDS}}(M)$ is the space of functions on the set of isomorphism classes of permitted bundles.

Proposition 2.4.1. *If M is a closed $(n-1)$-manifold, then the trivial bundle $P_{\mathrm{triv}}\to M$ is permitted if and only if $\chi(M)$ is even.*

Proof. The action for P_{triv} and $\phi\in\mathrm{Aut}(P_{\mathrm{triv}})$ is

$$\left\langle\frac{w(M)\alpha((P_{\mathrm{triv}})_\phi)}{1+\alpha((P_{\mathrm{triv}})_\phi)},[S^1\times M]\right\rangle=\left\langle\frac{w(M)(x[\phi]+\alpha(P_{\mathrm{triv}}))}{1+(x[\phi]+\alpha(P_{\mathrm{triv}}))},[S^1\times M]\right\rangle\tag{2.4.2}$$

by (2.3.33). Since P_{triv} is trivial, $\alpha(P_{\mathrm{triv}})=0$, so

$$=\left\langle\frac{w(M)x[\phi]}{1+x[\phi]},[S^1\times M]\right\rangle.\tag{2.4.3}$$

Since $(x[\phi])^2\in H^2(S^1;\mathbb{Z}/2)=0$,

$$=\langle w(M)x[\phi],[S^1\times M]\rangle,\tag{2.4.4}$$

which by a Fubini theorem is

$$=\langle x[\phi],[S^1]\rangle\langle w(M),[M]\rangle.\tag{2.4.5}$$

If ϕ is nontrivial, $\langle x[\phi],[S^1]\rangle=1$. Hence the action is zero for all ϕ if and only if $\langle w(M),[M]\rangle$, which is $\chi(M)$ mod 2, vanishes. $\square$

Corollary 2.4.6. *Let M be simply connected. Then,*

$$Z_{\mathrm{GDS}}(M) \cong \begin{cases} 0, & \chi(M) \text{ odd} \\ \mathbb{C}, & \chi(M) \text{ even}. \end{cases} \tag{2.4.7}$$

Proof. All principal $\mathbb{Z}/2$-bundles over such a manifold are trivial, so we just have to check whether the trivial bundle is permitted. $\square$

It is worth comparing this to the α^n Dijkgraaf-Witten theory.

Lemma 2.4.8. *If $n > 1$ and M is a closed $(n-1)$-manifold, $Z^{\mathrm{cl}}_{\alpha^n}(S^1 \times M, (P_{\mathrm{triv}})_\phi) = 0$ for any automorphism ϕ. In particular, if M is simply connected, $Z_{\alpha^n}(M) \cong \mathbb{C}$.*

Proof. Let $\phi \in \mathrm{Aut}(P_{\mathrm{triv}})$, so

$$\alpha((P_{\mathrm{triv}})_\phi) = \alpha(P_{\mathrm{triv}}) + x[\phi] = x[\phi]. \tag{2.4.9}$$

The action is

$$\langle \alpha(P_\phi)^n, [S^1 \times M] \rangle = \langle (x[\phi])^n, [S^1 \times M] \rangle = 0. \tag{2.4.10}$$

$\square$

Proposition 2.4.11.

$$Z_{\mathrm{GDS}}(\mathbb{CP}^n \times \mathbb{RP}^2) \cong \begin{cases} \mathbb{C}, & n \text{ even} \\ \mathbb{C}^2, & n \text{ odd}. \end{cases} \tag{2.4.12}$$

Proof. Let $X := \mathbb{CP}^n \times \mathbb{RP}^2$, and let z be the generator of $H^1(X; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Since

$$\chi(X) = \chi(\mathbb{CP}^n)\chi(\mathbb{RP}^2) = \begin{cases} 0 \bmod 2, & n \text{ odd} \\ 1 \bmod 2, & n \text{ even}, \end{cases} \tag{2.4.13}$$

then by Proposition 2.4.1, the trivial bundle is permitted if and only if n is odd.

The other isomorphism class of principal $\mathbb{Z}/2$-bundles on X is the one whose total space is the universal cover of X, which we denote P. Then $\alpha(P) = z$, and for $\phi \in \mathrm{Aut}(P)$, the Lagrangian for $S^1 \times X$ and P_ϕ is

$$\frac{\alpha(P_\phi) w(S^1 \times X)}{1 + \alpha(P_\phi)} = \frac{(z + x[\phi]) w(\mathbb{RP}^2) w(\mathbb{CP}^n)}{1 + z + x[\phi]}. \tag{2.4.14}$$

Since $z + x[\phi]$ is nilpotent, $1 + z + x[\phi]$ is invertible, so

$$= \frac{(z + x[\phi])w(\mathbb{RP}^2)w(\mathbb{CP}^n)(1 + z + x[\phi])}{(1 + z + x[\phi])^2}. \tag{2.4.15}$$

Since $(x[\phi])^2 = 0$,

$$= \frac{(1+z)^3(z + z^2 + x[\phi])w(\mathbb{CP}^n)}{1 + z^2} \tag{2.4.16}$$

$$= (1+z)(z + z^2 + x[\phi])w(\mathbb{CP}^n) \tag{2.4.17}$$

$$= (z + x[\phi] + zx[\phi])w(\mathbb{CP}^n). \tag{2.4.18}$$

We want to pair this with $[S^1 \times X]$, but (2.4.18) has no terms of degree $\dim(S^1 \times X) = 2n + 3$. Thus

$$\langle (z + x[\phi] + zx[\phi])w(\mathbb{CP}^n), [S^1 \times X] \rangle = 0, \tag{2.4.19}$$

so this bundle is always permitted. $\square$

Proposition 2.4.20. *For $n \geq 2$,*

$$Z_{\mathrm{GDS}}(\mathbb{RP}^n) \cong \begin{cases} \mathbb{C}, & n \text{ even} \\ \mathbb{C}^2, & n \text{ odd}. \end{cases} \tag{2.4.21}$$

Proof. Let $z \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$ denote the generator. By Proposition 2.4.1, the trivial principal $\mathbb{Z}/2$-bundle is permitted if and only if n is odd. The other isomorphism class of principal $\mathbb{Z}/2$-bundles is the universal cover $S^n \twoheadrightarrow \mathbb{RP}^n$, with $\alpha(S^n) = z$, so it suffices to prove this bundle is always permitted. Let ϕ be an automorphism of this principal bundle. The action is

$$\frac{\alpha(S^n_\phi)w(\mathbb{RP}^n)}{1 + \alpha(S^n_\phi)} = \frac{(z + x[\phi])(1+z)^{n+1}}{1 + z + x[\phi]}. \tag{2.4.22}$$

Again, $z + x[\phi]$ is nilpotent, so $1 + z + x[\phi]$ is invertible, so

$$= \frac{(z + x[\phi])(1+z)^{n+1}(1 + z + x[\phi])}{(1 + z + x[\phi])^2} \tag{2.4.23}$$

$$= \frac{(1+z)^{n+1}(z + z^2 + x[\phi])}{(1+z)^2} \tag{2.4.24}$$

$$= (1+z)^{n-1}(z + z^2 + x[\phi]). \tag{2.4.25}$$

But in (2.4.25), only the $(1+z)^{n-1}z^2$ term contributes anything of degree $\dim(S^1\times\mathbb{RP}^n)=n+1$, and this lives in $H^{n+1}(\mathbb{RP}^n;\mathbb{Z}/2)\otimes H^0(S^1;\mathbb{Z}/2)$, hence must be 0. Thus (2.4.25) has no terms of top degree, so

$$\langle(1+z)^{n+1}(z+z^2+x[\phi]),[S^1\times\mathbb{RP}^n]\rangle=0, \tag{2.4.26}$$

and this bundle is always permitted. ☐

We now compare Z_{GDS} with $\mathbb{Z}/2$-Dijkgraaf-Witten theories.

Lemma 2.4.27. *Let M be a closed $(2k+1)$-manifold and $y\in H^1(M;\mathbb{Z}/2)$. Then $w_1(M)y^{2k}=0$.*

Proof. Let v_1 denote the first Wu class. Then,

$$w_1y^{2k}=v_1y^{2k}=\mathrm{Sq}^1((y^k)^2)=0. \tag{2.4.28}$$

☐

Theorem 2.4.29. *In dimension 3, Z_{GDS} is isomorphic to Z_{α^3}.*

Proof. This follows from Proposition 2.2.32 after observing

$$(\alpha+w_1)^3=\alpha^3+w_1\alpha^2+w_1^2\alpha+w_1^3. \tag{2.4.30}$$

On any closed 3-manifold, $w_1^3=0$ because all closed 3-manifolds bound, and $w_1\alpha^2=0$ by Lemma 2.4.27. Thus (2.4.30) agrees with the Lagrangian for Z_{GDS}. ☐

The relationship in dimension 3 between the double semion model and the $\mathbb{Z}/2$-Dijkgraaf-Witten theory with Lagrangian α^3 is known to physicists (see, e.g., [**WW15**, §II]), though not previously proven in this form.

Theorem 2.4.31. *For even n, Z_{GDS} is isomorphic to DW_0.*

Proof. By Corollary 2.2.26, it suffices to prove that $w(M)\alpha/(1+\alpha)=0$ for any even-dimensional manifold M and $\alpha\in H^1(M;\mathbb{Z}/2)$. In Proposition 2.3.49, we saw $\langle w(M)\alpha/(1+\alpha),[M]\rangle$ is the mod 2 Euler characteristic of a submanifold N representing the Poincaré dual of α. Since N is a closed, odd-dimensional manifold, its mod 2 Euler characteristic vanishes, so $w(M)\alpha/(1+\alpha)=0$. ☐

[**FH16b**, Thm. 5.3] proved this for state spaces, and the proof idea is the same.

Theorem 2.4.32. *For odd $n\geq 4$, Z_{GDS} is not isomorphic to any $\mathbb{Z}/2$-Dijkgraaf-Witten theory.*

Proof. By Corollary 2.2.28, it suffices to prove that Z_{GDS} is not isomorphic to DW_0 and Z_{α^n}.

If $n = 4k+1$ for some $k \geq 1$, then $Z_{\mathrm{GDS}}(\mathbb{CP}^{2k}) = 0$ by Corollary 2.4.6, but $\mathrm{DW}_0(\mathbb{CP}^{2k}) \cong \mathbb{C}$, and $Z_{\alpha^n}(\mathbb{CP}^{2k}) \cong \mathbb{C}$ by Lemma 2.4.8.

If $n = 4k+3$ for some $k \geq 1$, then $Z_{\mathrm{GDS}}(\mathbb{CP}^{2k} \times \mathbb{RP}^2) \cong \mathbb{C}$ by Proposition 2.4.11 and $\mathrm{DW}_0(\mathbb{CP}^{2k} \times \mathbb{RP}^2) \cong \mathbb{C}^2$. For the theory with Lagrangian α^n, Lemma 2.4.8 gives us one copy of $\mathbb{C}$ from the trivial bundle. If $P \to \mathbb{CP}^{2k} \times \mathbb{RP}^2$ denotes the nontrivial bundle and $z \in H^1(\mathbb{RP}^2; \mathbb{Z}/2)$ denotes the generator, then $\alpha(P) = z$. For any $\phi \in \mathrm{Aut}(P)$,

$$\langle \alpha(P_\phi)^n, [S^1 \times \mathbb{CP}^{2k} \times \mathbb{RP}^2] \rangle = \langle (z + x[\phi])^n, [S^1 \times \mathbb{CP}^{2k} \times \mathbb{RP}^2] \rangle. \tag{2.4.33}$$

Since $(x[\phi])^2 = 0$, this is

$$= \langle z^n + nz^{n-1}x[\phi], [S^1 \times \mathbb{CP}^{2k} \times \mathbb{RP}^2] \rangle, \tag{2.4.34}$$

and since $z^3 = 0$, this is 0. Thus the state space picks up another factor of $\mathbb{C}$, and $Z_{\alpha^n}(\mathbb{CP}^{2k} \times \mathbb{RP}^2) \cong \mathbb{C}^2$. $\square$

This was also proven in [**FH16b**, Thm. 8.1], with the same manifolds as counterexamples.

CHAPTER 3

The Arf-Brown TFT of pin$^-$ manifolds

The content of this chapter is joint with Sam Gunningham and was published as [**DG18**], and is used here with his permission. It has been lightly edited to be streamlined with the rest of the thesis. Both authors contributed equally to the work, and worked together on all parts of the project.

3.0. Introduction

As part of a general program to classify and understand topological phases of matter within condensed-matter physics, there is a large body of recent work focusing on the special case of symmetry-protected topological (SPT) phases. This classification question has been studied by many authors in different settings and with many different approaches: for lists of references, see [**GJF19**, §1] and [**FH16a**, §9.3]. It is believed that the low-energy physics of SPT phases is often described by invertible topological quantum field theories (TFTs), which admit a purely mathematical classification, and that the classification of a given class of SPTs often agrees with the classification of the analogous class of invertible TFTs. At the same time, work on the mathematical theory of invertible TFTs has understood their classification as a problem in stable homotopy theory [**GMTW09, FH16a, SP17**]. Freed-Hopkins [**FH16a**] use this to answer the classification problem across a wide range of dimensions and symmetry types.

In this paper, we explain this perspective on classifying invertible TFTs and SPT phases in a specific setting, focusing on 2-dimensional theories formulated on manifolds with a pin$^-$ structure. Freed-Hopkins show that the group of deformation classes of 2d invertible pin$^-$ TFTs is isomorphic to $\mathbb{Z}/8$, and is generated by a TFT Z_{AB} whose partition function is the Arf-Brown invariant of a pin$^-$ surface, a generalization of the Arf invariant of a spin surface.

In §3.2, we provide three definitions for the Arf-Brown invariant, and compare each to an analogous definition of the Arf invariant: in §3.2.1, the original intersection-theoretic description due to Brown [**Bro71**]; in §3.2.2, an index-theoretic description due to Zhang [**Zha94, Zha17**]; and in §3.2.3, a new description using a twisted Atiyah-Bott-Shapiro map.

Then, in §3.3, we discuss how the classification of 2d invertible TFTs reduces to a homotopy-theoretic problem. Our approach follows Freed-Hopkins [**FH16a**, §5], but the fact that we're in dimension 2 allows for an explicit description of the 2-categories and homotopy 2-types that enter this argument, which are more complicated in higher dimensions. Moreover, some aspects of the story, such as the choice of a target category and the stable homotopy hypothesis, are understood in dimension 2 but not in higher dimensions. In §3.3.1, we review some generalities of invertible 2d TFTs, and in §3.3.2 discuss the stable homotopy hypothesis in dimension 2, a theorem of Gurski-Johnson-Osorno [**GJO19**]. We use this in §3.3.3 to classify 2d invertible TFTs of a given symmetry type valued in the Morita 2-category $\mathsf{sAlg}_{\mathbb{C}}$ of complex superalgebras. This is similar to a classification theorem of Schommer-Pries [**SP17**, Theorem 7.6] but with a different choice of target; hence in Proposition 3.3.22, we provide a proof for a folklore theorem identifying the homotopy type of $\mathsf{sAlg}_{\mathbb{C}}^{\times}$.

In §3.4, we apply this to pin$^-$ theories: the homotopy-theoretic approach to invertible TFTs and the KO-theoretic description of the Arf-Brown invariant combine to define the Arf-Brown TFT Z_{AB}, which was previously known to exist but hasn't been explicitly studied until now. We discuss what this theory assigns to closed pin$^-$ 0-, 1-, and 2-manifolds and how it relates to the twisted Atiyah-Bott-Shapiro orientation.

Finally, in §3.5, we discuss a conjectural appearance of the Arf-Brown TFT in physics, as the low-energy theory of the Majorana chain. In §3.5.1, we provide some background on SPTs and the low-energy approach to their classification. We then formulate the Majorana chain on an arbitrary compact pin$^-$ 1-manifold in §3.5.3, and discuss its low-energy TFT and how it relates to the Arf-Brown TFT in §3.5.4. We find that the space of ground states of the Majorana chain depends on a pin$^-$ structure, which is expected, but doesn't appear to have been determined before.

We also provide some preliminaries on Clifford algebras and pin manifolds in §3.1.1, and on the stable homotopy theory that we use in §3.1.2.

3.1. Preliminaries

3.1.1. Clifford algebras, pin groups, and pin structures. Pin structures are generalizations of spin structures to unoriented vector bundles and manifolds. In this section, we define the pin groups and state a few useful results about them. For proofs and a more detailed exposition, see [**ABS64**].

Definition 3.1.1. Let k be a field of characteristic not equal to 2, S be a finite set, and $\mathfrak{o}\colon S \to \{\pm 1\}$ be a function. The *Clifford algebra* $C\ell(k, S, \mathfrak{o})$ is defined to be the k-algebra

$$C\ell(k, S, \mathfrak{o}) \coloneqq T(k[S])/(s^2 = \mathfrak{o}(s), st = -ts \mid s, t \in S, s \neq t), \tag{3.1.2}$$

where $T(k[S])$ denotes the tensor algebra of the space of functions $S \to k$, and we identify s with the function equal to 1 at s and 0 elsewhere.

For $S \coloneqq \{1, \ldots, m\} \cup \{-1, \ldots, -n\}$ and $\mathfrak{o}(x) \coloneqq \mathrm{sign}(x)$, we'll write $C\ell_{m,n}(k) \coloneqq C\ell(k, S, \mathfrak{o})$, as well as $C\ell_n(k) \coloneqq C\ell_{n,0}(k)$ and $C\ell_{-n}(k) \coloneqq C\ell_{0,n}(k)$. If $k = \mathbb{C}$, we'll suppress $\mathbb{C}$ from the notation, e.g. writing $C\ell_{m,n}$, $C\ell_n$, and $C\ell_{-n}$.

The ideal in the quotient in (3.1.2) contains only even-degree elements of the tensor algebra, so the Clifford algebras are $\mathbb{Z}/2$-graded algebras, or *superalgebras*. If a is a homogeneous element in a $\mathbb{Z}/2$-graded algebra or module, we will let $|a| \in \mathbb{Z}/2$ denote its degree.

Lemma 3.1.3 ([**ABS64**, Proposition 1.6])**.** *Let S_1 and S_2 be finite sets and $\mathfrak{o}_i \colon S_i \to \{\pm 1\}$ be functions. If $\mathfrak{o} \colon S_1 \amalg S_2 \to \{\pm 1\}$ is $\mathfrak{o}_i$ on S_i, then there is a canonical isomorphism*

$$C\ell(k, S_1, \mathfrak{o}_1) \otimes_k C\ell(k, S_2, \mathfrak{o}_2) \cong C\ell(k, S_1 \amalg S_2, \mathfrak{o}).$$

For this to be true, we must use the graded tensor product, whose multiplication contains a sign: if a, b, a', b' are homogeneous elements, then

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b'||a'|} aa' \otimes bb'. \tag{3.1.4}$$

Let $\alpha \in \mathrm{End}(C\ell(k, S, \mathfrak{o}))$ be the *grading operator*, whose action on a homogeneous element a is multiplication by $(-1)^{|a|}$.

Definition 3.1.5. The *Clifford group* is

$$\Gamma(k, S, \mathfrak{o}) \coloneqq \{x \in C\ell(k, S, \mathfrak{o})^\times \mid \alpha(x) y x^{-1} \in k[S] \subset C\ell(k, S, \mathfrak{o}) \text{ for all } y \in k[S]\}.$$

Here we use the canonical map $k[S] \hookrightarrow T(k[S]) \twoheadrightarrow C\ell(k, S, \mathfrak{o})$, which is injective.

Definition 3.1.6. There is an involution $\beta \colon C\ell(k, S, \mathfrak{o}) \to C\ell(k, S, \mathfrak{o})$ induced from the map $\widetilde{\beta} \colon T(k[S]) \to T(k[S])$ sending a homogeneous element

$$f_1 \otimes \cdots \otimes f_n \mapsto f_n \otimes \cdots \otimes f_1.$$

The *Clifford norm* $N \colon \Gamma(k, S, \mathfrak{o}) \to k^\times$ is defined by $N(x) \coloneqq \beta(x) \cdot x$.

Definition 3.1.7. The *pin group* $\mathrm{Pin}(k, S, \mathfrak{o})$ associated to the Clifford algebra in Definition 3.1.1 is the kernel of the Clifford norm. The *spin group* $\mathrm{Spin}(k, S, \mathfrak{o})$ is the subgroup of $\mathrm{Pin}(k, S, \mathfrak{o})$ which is even in the grading on the Clifford algebra.

We are interested in the case where $k = \mathbb{R}$, so that the pin and spin groups are Lie groups. If we specialize to $C\ell_{\pm n}(\mathbb{R})$, they're compact Lie groups.

Definition 3.1.8. Let Pin_n^+ denote the pin group associated to $C\ell_n(\mathbb{R})$ and Pin_n^- denote the pin group associated to $C\ell_{-n}(\mathbb{R})$. The corresponding spin groups are canonically isomorphic, so we denote either one by Spin_n.

Proposition 3.1.9. *Let* $\mathrm{Pin}_n^\pm$ *denote either of* Pin_n^+ *or* Pin_n^-*. Then, there are group extensions*

$$1 \longrightarrow \mathrm{Spin}_n \longrightarrow \mathrm{Pin}_n^\pm \xrightarrow{\pi_0} \mathbb{Z}/2 \longrightarrow 1 \tag{3.1.10a}$$

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Pin}_n^\pm \xrightarrow{\rho} \mathrm{O}_n \longrightarrow 1. \tag{3.1.10b}$$

Let $\rho\colon H \to G$ be a homomorphism of Lie groups and $\pi\colon P \to M$ be a principal G-bundle. Recall that a *reduction of the structure group* of P to H is data $(\pi'\colon Q \to M, \theta)$ such that

- $\pi'\colon Q \to M$ is a principal H-bundle, and
- $\theta\colon Q \times_H G \to P$ is an isomorphism of principal G-bundles, where H acts on G through ρ.

An equivalence of reductions $(Q_1, \theta_1) \to (Q_2, \theta_2)$ is a map $\psi\colon Q_1 \to Q_2$ intertwining θ_1 and θ_2.

Definition 3.1.11. If $\rho\colon H \to \mathrm{GL}_n(\mathbb{R})$ is a Lie group homomorphism, an *H-structure* on a vector bundle $E \to X$ is an equivalence class of reductions of the structure group of the principal $\mathrm{GL}_n(\mathbb{R})$-bundle of frames of E to H. If M is a smooth manifold and $E = TM$, this is called a *tangential H-structure* on M; if M is a smooth manifold and E is its stable normal bundle, this is called a *normal H-structure*.

For example, an SO_n-structure is the same thing as an orientation. A *spin structure* on an n-manifold M is a tangential Spin_n structure, and we define pin^+ and pin^- structures analogously.

Remark 3.1.12. We note that such structures are *stable* in the following sense: a (s)pin$^\pm$ structure on a vector bundle V is equivalent to a (s)pin$^\pm$ structure on $V \oplus \underline{\mathbb{R}}$. In particular, a stable framing on a vector bundle (a trivialization of $V \oplus \underline{\mathbb{R}}^N$ for some N) determines a (s)pin$^\pm$ structure.

Proposition 3.1.13 ([**KT90b**, Lemma 1.3])**.** *Let $E \to X$ be a vector bundle and $w_n(E) \in H^n(X; \mathbb{Z}/2)$ denote its n^{th} Stiefel-Whitney class.*

- *E admits a spin structure iff $w_1(E) = 0$ and $w_2(E) = 0$.*
- *E admits a pin$^+$ structure iff $w_2(E) = 0$.*
- *E admits a pin$^-$ structure iff $w_2(E) + w_1(E)^2 = 0$.*

In all cases, if E admits one of these structures, the set of such structures (in the spin case, with fixed orientation) on E is an $H^1(X; \mathbb{Z}/2)$-torsor.

Corollary 3.1.14. *Let M be a closed manifold of dimension at most 2. Then, M has a pin$^-$ structure, and has a spin structure if and only if it is orientable. If $\dim M = 2$, then M has a pin$^+$ structure iff its Euler characteristic is even.*

Remark 3.1.15. There are a few facts about pin structures which might be surprising to a reader who has only studied spin manifolds. A tangential spin structure is equivalent data to a normal spin structure, but this is false for pin structures: a tangential pin$^+$ structure is equivalent to a normal pin$^-$ structure, and vice versa. This is discussed in [**KT90b**, §1], and will be relevant in our homotopical approach to 2D pin$^-$ TFTs.

The product of spin manifolds has an induced spin structure, but using Proposition 3.1.13 one can write down pin$^-$ manifolds whose product doesn't have a pin$^-$ structure; the same phenomenon occurs for pin$^+$ structure. This means that the pin$^+$ and pin$^-$ bordism groups are not rings, though they are modules over the spin bordism ring. See also Lemma 1.1.17 for a homotopical lift of this statement.

Proposition 3.1.16 ([**KT90b**, §2])**.** *Let Ω_n^H denote the bordism group of n-manifolds with tangential H-structure. Then:*

(1) There are isomorphisms $\Omega_1^{\mathrm{Spin}} \cong \mathbb{Z}/2$ and $\Omega_1^{\mathrm{Pin}^-} \cong \mathbb{Z}/2$, and the forgetful map $\Omega_1^{\mathrm{Spin}} \to \Omega_1^{\mathrm{Pin}^-}$ is an isomorphism. Both are generated by the circle with structure induced by its Lie group framing.

(2) There are isomorphisms $\Omega_2^{\mathrm{Spin}} \cong \mathbb{Z}/2$ and $\Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ which identify the forgetful map $\Omega_2^{\mathrm{Spin}} \to \Omega_2^{\mathrm{Pin}^-}$ with the map sending $1 \mapsto 4$. The torus with spin structure induced from its Lie group framing generates Ω_2^{Spin}, and $\mathbb{RP}^2$ (with either of its two pin$^-$ structures) generates $\Omega_2^{\mathrm{Pin}^-}$.

In particular, the two isomorphism classes of pin$^-$ circles aren't cobordant: one bounds and the other doesn't. We denote the bounding pin$^-$ circle by S^1_b, and the nonbounding pin$^-$ circle by S^1_{nb}. This applies *mutatis mutandis* to the two spin circles.

3.1.2. Homotopy theory. We follow the conventions in [**BC18**]. If the reader is unfamiliar with spectra and the stable homotopy category we recommend they first read Section 2 of loc. cit. Here, we briefly recall some notation, basic definitions, and examples.

3.1.2.1. *The (stable) homotopy category.*

- The unstable homotopy category is denoted $h\mathcal{S}$. This category receives a map from the category Top of topological spaces and continuous maps which takes weak equivalences in Top to isomorphisms in $h\mathcal{S}$; by abuse of notation we will denote the image of a topological space in $h\mathcal{S}$ by the same symbol, and refer to the objects of $h\mathcal{S}$ as spaces.
- We write $[X, Y]$ for the space of morphisms in $h\mathcal{S}$ between spaces X and Y; if we choose "nice enough" representatives for X and Y (for example CW-complexes), then this is given by the set of homotopy classes of maps between X and Y.
- We denote by $h\mathcal{S}p$ the stable homotopy category (also known as the homotopy category of spectra). This category receives a functor from the category of prespectra which takes weak equivalences of prespectra to isomorphisms in $h\mathcal{S}p$. As in the unstable case, we will refer to objects of $h\mathcal{S}p$ simply as spectra, and use the same symbol for a prespectrum and its corresponding object in $h\mathcal{S}p$.
- Given a pair of spectra E, F, we write $[E, F]$ for the set of morphisms $\mathrm{Hom}_{h\mathcal{S}p}(E, F)$, and $[E, F]_n$ for $[\Sigma^n E, F]$; if E and F are "nice enough" (for example, if they are *CW-spectra* – prespectra such that each space is a CW complex, the structure maps are cellular inclusions, and the adjoints of the structure maps are homeomorphisms) then $[E, F]$ is given by homotopy classes of maps between spectra. There are natural abelian group structures on $[E, F]$ and $[E, F]_n$.
- The category $h\mathcal{S}p$ carries a symmetric monoidal structure $\wedge$, called the smash product. There is also a mapping object (internal hom) $\mathrm{Map}(E, F)$ whose homotopy groups are $\pi_n(\mathrm{Map}(E, F)) = [E, F]_n$.

3.1.2.2. *Examples of spectra.*

Example 3.1.17. Given a pointed space X, we have the *suspension spectrum* $\Sigma^\infty X$, which may be presented by a prespectrum whose nth space is $\Sigma^n X$. A special case of this construction is the *sphere spectrum* $\mathbb{S} := \Sigma^\infty S^0$, which is the unit object for the smash product.

Given a spectrum E and a space X, we write $E^n(X)$ for the *E-cohomology group* $[\Sigma^\infty X, E]_n$. Similarly, we write $E_n(X)$ for the *E-homology group* $\pi_n(\Sigma^\infty X \wedge E)$. These are examples of *generalized cohomology* (resp. *homology*) *theories*: they satisfy all of the Eilenberg-Steenrod axioms except the dimension axiom.

Remark 3.1.18. The Brown representability theorem [**Bro62**] states that any generalized cohomology theory h^* (resp. generalized homology theory h_*) arises from a spectrum E in this manner; we say that h^* (resp. h_*) is *represented by E*.

Example 3.1.19. Ordinary cohomology with coefficients in an abelian group A is represented by the *Eilenberg-MacLane spectrum* HA. This may be modeled as a spectrum whose nth space is the Eilenberg-MacLane space $K(n, A)$ for $n > 0$, and whose nonpositive spaces are trivial.

Complex K-theory is a generalized cohomology theory represented by a spectrum denoted KU. Similarly, real K-theory is represented by a spectrum KO.

A spectrum is called *connective* if it has trivial negative homotopy groups. Given a connective spectrum E, its zeroth space[1] has the structure of an infinite loop space. In fact, the homotopy theory of connective spectra is equivalent to that of infinite loop spaces: given an infinite loop space, by definition it has a sequence of deloopings which form the spaces in the corresponding spectrum. See Adams [**Ada78**] for more on this correspondence.

In this chapter, we also need Thom spectra, as defined in §1.1.2; specifically, given $\xi\colon B \to B\mathrm{O}$, recall from Definition 1.1.12 the *Madsen-Tillmann spectra* $MT\xi_n$, $MT\xi$, which are defined as the Thom spectra of the inverse of the tautological bundles over $B_n := B \times_{B\mathrm{O}} B\mathrm{O}_n$, resp. B; as well as the Thom spectra $M\xi_n$, and $M\xi$ of those tautological bundles. The homotopy groups of $MT\xi$ compute bordism groups of manifolds with a tangential ξ-structure, and the homotopy groups of $M\xi$ compute bordism groups of manifolds with a normal ξ-structure. Thom spectra are also useful for understanding duality in $h\mathcal{S}p$.

Theorem 3.1.20 (Atiyah [**Ati61b**])**.** *Let M be a closed manifold. Then $\Sigma^\infty M$ is dualizable in $h\mathcal{S}p$, and its dual is the Thom spectrum of the stable normal bundle ν of M.*

We won't say very much about duality, but we note in particular that if B is a spectrum with dual $B^\vee$, then for any spectra A and C there is a weak equivalence

$$\mathrm{Map}(A \wedge B, C) \simeq \mathrm{Map}(A, B^\vee \wedge C). \tag{3.1.21}$$

For more on duality, see [**Ada74**, §III.5].

3.2. The Arf-Brown invariant of a pin$^-$ surface

In this section, we give various constructions of the Arf-Brown invariant of a pin$^-$ surface: intersection theoretic in §3.2.1, index-theoretic in §3.2.2, and KO-theoretic in §3.2.3.

[1] Here it is essential that one considers a spectrum rather than just a prespectrum (i.e. the adjoints of the structure maps are homeomorphisms).

3.2.1. Intersection-theoretic descriptions of the invariants. The Arf invariant of a spin surface and the Arf-Brown invariant of a pin$^-$ surface are complete bordism invariants defined using intersection theory.

3.2.1.1. *The Arf invariant of a spin surface.* Let Σ be a closed oriented surface. If $x, y \in H_1(\Sigma;\mathbb{Z}/2)$, then the mod 2 intersection number $I_2(x,y) \in \mathbb{Z}/2$ is defined by choosing smooth, transverse representative curves for x and y and computing the number of points mod 2 in their intersection. This does not depend on the choice of representatives and defines a non-degenerate bilinear pairing

$$I_2\colon H_1(\Sigma;\mathbb{Z}/2) \otimes H_1(\Sigma;\mathbb{Z}/2) \to \mathbb{Z}/2.$$

A $\mathbb{Z}/2$-*quadratic enhancement* of I_2 is a quadratic form on $H_1(\Sigma;\mathbb{Z}/2)$ whose induced bilinear form is I_2. Explicitly, this is a function

$$q\colon H_1(\Sigma;\mathbb{Z}/2) \to \mathbb{Z}/2$$

such that for all $x, y \in H_1(\Sigma;\mathbb{Z}/2)$,

$$q(x+y) = q(x) + q(y) + I_2(x,y). \tag{3.2.1}$$

The set of $\mathbb{Z}/2$-quadratic enhancements of I_2 is an $H^1(\Sigma;\mathbb{Z}/2)$-torsor: given a $\gamma \in H^1(\Sigma;\mathbb{Z}/2)$ and a quadratic enhancement $q\colon H_1(\Sigma;\mathbb{Z}/2) \to \mathbb{Z}/2$, the function $q_\gamma(x) \coloneqq q(x) + \gamma(x)$ is again a quadratic enhancement.

We have the following relationship between spin structures and quadratic enhancements of the intersection form.

Theorem 3.2.2 ([**Joh80**, **Ati71**])**.** *There is an isomorphism of $H^1(\Sigma;\mathbb{Z}/2)$-torsors between the set of $\mathbb{Z}/2$-quadratic enhancements of I_2 and isomorphism classes of spin structures on Σ.*

Remark 3.2.3. Given a spin structure on Σ, the associated quadratic form is easy to describe: it takes a homology class represented by an embedded circle to either 0 or 1 depending on whether the induced spin structure on the circle is bounding or non-bounding.

Definition 3.2.4. Given a spin surface Σ with corresponding quadratic form q, the Arf invariant of q may be defined as follows. If $\{e_i, f_i\}$ is a basis of $H_1(\Sigma;\mathbb{Z}/2)$ which is symplectic with respect to the intersection form, then

$$\mathrm{Arf}(\Sigma) \coloneqq \sum_i q(e_i)q(f_i) \in \mathbb{Z}/2. \tag{3.2.5}$$

Theorem 3.2.6 ([**KT90b**])**.** *The Arf invariant is a spin bordism invariant, and defines an isomorphism*

$$\mathrm{Arf} : \Omega_2^{\mathrm{Spin}} \to \mathbb{Z}/2. \tag{3.2.7}$$

Example 3.2.8. Let $T = S^1 \times S^1$ denote the torus with spin structure afforded by the Lie group framing. Consider the symplectic basis $\{e, f\}$ for $H_1(T; \mathbb{Z}/2)$ corresponding to the embedded circles $S^1 \times \{1\}$ and $\{1\} \times S^1$. As each circle carries the non-bounding spin structure, the associated quadratic form q takes the values $q(e) = q(f) = 1$. Thus the Arf invariant is $1 \in \mathbb{Z}/2$, and hence T is a generator for the spin bordism group.

3.2.1.2. *The Arf-Brown invariant of a pin$^-$ surface.* Now suppose Σ is any closed surface (not necessarily oriented). Then $H_1(\Sigma; \mathbb{Z}/2)$ still carries a non-degenerate intersection form I_2, although $H_1(\Sigma; \mathbb{Z}/2)$ may be odd dimensional, and will not admit a symplectic basis in general. In this case, one must consider the following notion:

Definition 3.2.9. A $\mathbb{Z}/4$-*quadratic enhancement of the intersection form* on Σ is a function

$$q \colon H_1(\Sigma; \mathbb{Z}/2) \to \mathbb{Z}/4 \tag{3.2.10}$$

such that for all $x, y \in H_1(\Sigma; \mathbb{Z}/2)$,

$$q(x + y) = q(x) + q(y) + 2 \cdot I_2(x, y), \tag{3.2.11}$$

where $(2\cdot) \colon \mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ is inclusion.

As with $\mathbb{Z}/2$-quadratic enhancements, the set of $\mathbb{Z}/4$-quadratic enhancements is an $H^1(\Sigma; \mathbb{Z}/2)$-torsor: given a $\gamma \in H^1(\Sigma; \mathbb{Z}/2)$ and a quadratic enhancement $q \colon H_1(\Sigma; \mathbb{Z}/2) \to \mathbb{Z}/4$, the function $q_\gamma(x) \coloneqq q(x) + 2 \cdot \gamma(x)$ is again a $\mathbb{Z}/4$-quadratic enhancement.

Theorem 3.2.12 ([**KT90b**])**.** *For any closed surface Σ, there is an isomorphism of $H^1(\Sigma; \mathbb{Z}/2)$-torsors from the set of pin$^-$ structures on Σ to the set of $\mathbb{Z}/4$-quadratic enhancements of the intersection form on Σ.*

Definition 3.2.13 ([**Bro71, KT90b**])**.** Let Σ be a pin$^-$ surface and let $q \colon H_1(\Sigma; \mathbb{Z}/2) \to \mathbb{Z}/4$ be its associated quadratic enhancement. The *Arf-Brown invariant* of Σ is the unit complex number

$$AB(\Sigma) \coloneqq \frac{1}{\sqrt{|H_1(\Sigma; \mathbb{Z}/2)|}} \sum_{x \in H_1(\Sigma; \mathbb{Z}/2)} \exp\left(\frac{2\pi i q(x)}{4}\right). \tag{3.2.14}$$

This is sometimes called the *Kervaire invariant* or the *Arf-Brown-Kervaire invariant.*

Theorem 3.2.15 ([**Bro71**, **KT90b**]). *The Arf-Brown invariant $AB(\Sigma)$ is a pin$^-$ bordism invariant, and defines an isomorphism*

$$AB : \Omega_2^{\mathrm{Pin}^-} \to \mu_8 \cong \mathbb{Z}/8 \tag{3.2.16}$$

where μ_8 denotes the group of eighth roots of unity.

Example 3.2.17. Let us compute the value of the Arf-Brown invariant for the pin$^-$ structures on $\mathbb{RP}^2$. In that case $H_1(\mathbb{RP}^2; \mathbb{Z}/2) \cong \mathbb{Z}/2$, and there are two quadratic enhancements of the intersection form which take the image of the non-zero homology class to either 1 or 3 mod 4. In the first case, we see that the Arf-Brown invariant is $\exp(\frac{2\pi i}{8})$, and in the second $\exp(\frac{-2\pi i}{8})$. It follows that either structure gives a generator for pin$^-$ bordism.

If Σ is an oriented surface, then a $\mathbb{Z}/4$-quadratic enhancement is necessarily valued in the even elements of $\mathbb{Z}/4$, and thus recovers the $\mathbb{Z}/2$-quadratic enhancement corresponding to a spin structure. Moreover, the Arf-Brown invariant of such a quadratic enhancement is the (exponentiated) Arf invariant of the corresponding quadratic form.

3.2.2. Index-theoretic description of the invariants. The Arf(-Brown) invariant of a spin (or pin$^-$) surface admits an alternative description in terms of Dirac operators acting on sections of (s)pinor bundles. In the spin case, the Arf invariant corresponds to the mod 2 index or Atiyah invariant of a spin Riemann surface – the mod 2 dimension of the space of holomorphic sections of a theta-characteristic. In the pin$^-$ case, the Arf-Brown invariant may be interpreted as the reduced η-invariant of a twisted Dirac operator as defined and studied by Zhang [**Zha94**, **Zha17**].

3.2.2.1. *The Atiyah invariant of a spin surface.* Let Σ be a closed surface equipped with a Riemannian metric and a spin structure.[2] Then Σ carries a graded spinor bundle

$$S_\Sigma = P_{\mathrm{Spin}_2} \times_{\mathrm{Spin}_2} C\ell_{-2}(\mathbb{R}) \tag{3.2.18}$$

with a left action of the bundle of Clifford algebras $C\ell(T^*_\Sigma)$ and a commuting right action of the constant algebra $C\ell_{-2}(\mathbb{R})$. This bundle splits as a sum of its graded components $S^0_\Sigma \oplus S^1_\Sigma$, where each S^i_Σ carries a fiberwise action of $C\ell^0_{-2}(\mathbb{R}) \cong \mathbb{C}$ and thus may be considered as a complex line bundle.

There is a Dirac operator

$$D^+_\Sigma : C^\infty(S^0_\Sigma) \to C^\infty(S^1_\Sigma)$$

[2] Here, we consider a Riemannian metric to induce a *negative* definite quadratic form on the fibers of T^*_Σ.

given by composing the canonical connection operator

$$\nabla : C^\infty(S_\Sigma) \to C^\infty(T^*_\Sigma \otimes S_\Sigma)$$

with the action of sections of T^*_Σ via Clifford multiplication.

Definition 3.2.19. The *Atiyah invariant* of the spin surface Σ is

$$\dim \ker(D^+_\Sigma) \bmod 2 \in \mathbb{Z}/2.$$

Remark 3.2.20. The Riemannian metric and orientation on Σ determine a complex structure, and the even spinor bundle S^0_Σ defines a square root of the holomorphic cotangent bundle (such a square root is known as a *theta characteristic*). The Dirac operator is identified with the $\overline{\partial}$ operator defining the holomorphic structure on S^0_Σ. Thus the Atiyah invariant of Σ is the mod 2 dimension of the space of holomorphic sections of S^+_Σ.

Proposition 3.2.21 ([**Joh80**]). *Given a closed spin surface Σ, the Atiyah invariant and Arf invariant coincide.*

3.2.2.2. *Reduced η-invariant of a pin$^-$ surface.* Given a pin$^-$ surface Σ, the pinor bundle

$$S_\Sigma = P_{\mathrm{Pin}_2^-} \times_{\mathrm{Pin}_2^-} C\ell_{-2}(\mathbb{R})$$

still makes sense, though it doesn't carry a natural $\mathbb{Z}/2$-grading. However, as Clifford multiplication is not Pin_2^--equivariant, the formula for the Dirac operator now defines a map on sections

$$D_\Sigma : C^\infty(S_\Sigma) \to C^\infty(S_\Sigma \otimes \delta),$$

where δ is the orientation bundle.

To get an operator acting on sections of the same bundle, we may apply the following trick (which is spelled out in [**Sto88**] in the analogous case of a pin$^+$ manifold of dimension 4 mod 8). The left regular action of $C\ell_{-2}(\mathbb{R}) \cong \mathbb{H}$ on itself extends to an action of $C\ell_{-3}(\mathbb{R}) \cong \mathbb{H} \oplus \mathbb{H}$; now compose the operator D_Σ with the action of $e_3 \in C\ell_{-3}(\mathbb{R})$ to get a self-adjoint operator $\widetilde{D}_\Sigma$ on sections S_Σ called the *twisted Dirac operator*.

The twisted Dirac operator has an associated η-function, defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$ by the formula

$$\eta_{\widetilde{D}}(s) = \sum_{\lambda \neq 0} \mathrm{sign}(\lambda) \dim E(\lambda) |\lambda|^{-s}$$

where $E(\lambda)$ is the eigenspace with eigenvalue λ. This function admits a meromorphic extension to $s = 0$, and thus we may define the *reduced η-invariant*:

$$\overline{\eta}(\widetilde{D}) = \frac{\dim\ker(\widetilde{D}) + \eta_{\widetilde{D}}(0)}{2} \bmod 2\mathbb{Z} \in \mathbb{R}/2\mathbb{Z}.$$

One can check (see [**Zha17**, Proposition 2.7]) that if $\Sigma = \partial M$ is the boundary of a pin$^-$ 3-manifold M, the reduced eta invariant of Σ is zero (i.e. the non-reduced invariant is an even integer). Thus for any pin$^-$ surface Σ, the above quantity is a well-defined bordism invariant (and in particular is independent of the choice of metric used to define it).

Proposition 3.2.22 ([**Zha94**, **Zha17**])**.** *Given a pin$^-$ surface Σ, the reduced η-invariant $\overline{\eta}(\widetilde{D}_\Sigma)$ is an element of $\mathbb{Z}[\frac{1}{4}]/2\mathbb{Z}$, and agrees with the Arf-Brown invariant of Σ under the isomorphism given by the exponential map $\mathbb{Z}[\frac{1}{4}]/2\mathbb{Z} \cong \mu_8 \subseteq \mathbb{C}^\times$.*

Remark 3.2.23. In the spin case, the contribution from the η-invariant vanishes (as the spectrum is symmetric), and we are just left with half the dimension of $\ker(D_\Sigma)$ (or equivalently the dimension of $\ker(D_\Sigma^+)$) mod 2.

3.2.3. KO-theoretic descriptions of the invariants. Here we explain how the analytic index-theoretic invariants of the previous section may be expressed topologically in terms of pushforwards in (twisted) KO-theory.

3.2.3.1. *The Atiyah-Bott-Shapiro orientation and pushforward maps in KO-theory.* Let $\pi : V \to X$ be a rank-k real vector bundle equipped with a spin structure, and let $\mathrm{Th}(X, V)$ denote its Thom space. The Clifford module construction of Atiyah-Bott-Shapiro [**ABS64**] determines a Thom isomorphism

$$KO^*(X) \xrightarrow{\cong} \widetilde{KO}^{*+k}(\mathrm{Th}(X, V)). \tag{3.2.24}$$

This isomorphism is given by multiplication by a Thom class $U \in \widetilde{KO}^n(\mathrm{Th}(X, V))$, which may be described as follows. The spin structure on V determines a graded spinor bundle $S_V = S_V^0 \oplus S_V^1$ (see (3.2.18)), which carries a left action of the Clifford bundle $C\ell(V)$. Pulling S_V back to the total space of V, we obtain a pair of bundles together with a homomorphism

$$\pi^*(S_V^1) \to \pi^*(S_V^0)$$

given by Clifford multiplication, which is an isomorphism away from the zero section. This defines an element of $KO(V, V \setminus 0) \cong \widetilde{KO}(\mathrm{Th}(X, V))$ which is the required Thom class.

Using the Thom isomorphism, we can define a pushforward for an n-dimensional spin manifold M. Choose an embedding $M \to \mathbb{R}^N$ for some large N, and let $\nu \to M$ be the normal bundle, which has rank $N-n$. Using the tubular neighborhood theorem to embed $\nu \hookrightarrow \mathbb{R}^N$, consider the Pontrjagin-Thom collapse map

$$PT_\nu : S^N = \mathbb{R}^N \cup \{\infty\} \to \mathrm{Th}(M, \nu)$$

which takes the complement of the tubular neighborhood in S^N to the basepoint of the Thom space.

Definition 3.2.25. The *pushforward map in KO-theory* for X is the composition

$$\pi_!^M \colon KO^*(M) \xrightarrow[(3.2.24)]{} \widetilde{KO}^{*+N-n}(\mathrm{Th}(M,\nu)) \xrightarrow{PT_\nu^*} \widetilde{KO}^{*+N-n}(S^N) \xrightarrow{s} KO^{*-n}(\mathrm{pt}), \tag{3.2.26}$$

where s is the suspension isomorphism.

One may check that this invariant does not depend on the choice of embedding for large enough N. Moreover, by considering the appropriate modification for manifolds with boundary, one can check that the association of $\pi_!^M(1_M) \in KO^{-n}(\mathrm{pt})$ to a closed spin n-manifold is a bordism invariant.

Remark 3.2.27. Another perspective on the pushforward in KO-theory is that a closed spin n-manifold M carries a fundamental class $[M] \in KO_n(M)$, Spanier-Whitehead dual to the Thom class in $u_M \in \widetilde{KO}^{-n}(\Sigma^{-N}\mathrm{Th}(M,\nu))$. We can then pushforward to get a class in $KO_n(\mathrm{pt}) = KO^{-n}(\mathrm{pt})$.

The collection of Thom classes given by the Atiyah-Bott-Shapiro orientation may be succinctly formulated as a morphism of spectra (in fact of E_∞-ring spectra [**Joa04, AHR10**])

$$\widehat{A} \colon \mathit{MSpin} \to KO. \tag{3.2.28}$$

The Pontrjagin-Thom construction can then be interpreted as a homomorphism (in fact, isomorphism) from the spin bordism groups Ω_n^{Spin} to the homotopy groups $\pi_n(\mathit{MSpin})$, as in Theorem 1.1.14; the induced map on homotopy groups

$$\Omega_n^{\mathrm{Spin}} \cong \pi_n(\mathit{MSpin}) \to \pi_n(KO) = KO^{-n}(\mathrm{pt})$$

takes the class of a closed spin n-manifold M to the pushforward $\pi_!^M(1_M)$.

With the pushforward in hand, we can give a description for the Arf invariant of a spin surface which is simpler, if less intuitive, than the one given in the previous section.

Proposition 3.2.29 ([**Ati71**]). *Let Σ be a spin surface. The Atiyah/Arf invariant of Σ is the pushforward of* 1*:*

$$A'(\Sigma) \coloneqq \pi_!^{\Sigma}(1) \in KO^{-2}(\mathrm{pt}) \cong \mathbb{Z}/2. \tag{3.2.30}$$

Remark 3.2.31. From this perspective, the Arf invariant is an example of a generalized characteristic number, specifically a *KO-Pontrjagin number* as constructed by Anderson-Brown-Peterson [**ABP66**].

3.2.3.2. *Twisted Thom isomorphism for pin$^-$ manifolds.* Next we discuss the generalization of this invariant to pin$^-$ surfaces. One issue is that pin$^-$ vector bundles are not oriented for KO, so it is not immediately clear how to define a pushforward.

The key idea is the observation that a pin$^\pm$ structure on a vector bundle $V \to X$ is equivalent to a spin structure on the virtual vector bundle $V \mp \mathrm{Det}(V)$ (see [**KT90b**]). In particular, we have a Thom class

$$U \in \widetilde{KO}^n(\mathrm{Th}(X, V \mp \mathrm{Det}(V))) \tag{3.2.32}$$

Now if M is a closed pin$^-$ n-manifold with an embedding in $\mathbb{R}^N$ for $N \gg 0$, its normal bundle ν is equipped with a pin$^+$ structure, so we have a corresponding Thom class in KO-theory of the Thom spectrum:

$$U \in \widetilde{KO}^{N-n-1}(M^{\nu-\delta}) = \widetilde{KO}^{N-n}(M^{\nu+1-\delta}). \tag{3.2.33}$$

Alternatively, by Spanier-Whitehead duality, there is a fundamental class $[M] \in \widetilde{KO}_n(M^{\delta-1})$.

These ideas are best understood from the perspective of twisted KO-theory.

Definition 3.2.34. Given a space X equipped with a map $w : X \to B\mathrm{O}_1$ (which we may think of as classifying either a double cover $X^w \to X$ or a real line bundle $L_w \to X$), we define the *twisted KO-cohomology*

$$KO^{w+n}(X) \coloneqq \widetilde{KO}^n(X^{L_w-1}). \tag{3.2.35a}$$

Similarly, we have the *twisted KO-homology*

$$KO_{w+n}(X) \coloneqq \widetilde{KO}_n(X^{1-L_w}). \tag{3.2.35b}$$

Remark 3.2.36. This construction is an example of the notion of a twisted generalized cohomology theory. This and other examples can be found in [**FHT11**] (e.g. see Example 2.28) and [**ABG10**].

3.2.3.3. *Pushforward in twisted KO-homology.* Using this language, we may now give a KO-theoretic construction of the Arf-Brown (or reduced η-invariant) of a pin$^-$ surface Σ. Let

$$w_1 : \Sigma \to B\mathrm{O}_1$$

denote the classifying map of the orientation line bundle. Then consider the pushforward

$$w_{1!}([\Sigma]) \in KO_{2+w}(B\mathrm{O}_1). \tag{3.2.37}$$

The group

$$KO_{2+w}(B\mathrm{O}_1) = KO_2(\Sigma^{-1}MO_1) \cong KO_3(\mathbb{RP}^\infty)$$

is isomorphic to the direct limit of cyclic 2-groups $\mathbb{Z}/2^\infty$, or equivalently, the collection of all 2^nth roots of unity in $\mathbb{C}^\times$. One may also apply the above construction for the connective theory ko, in which case (see [**BG10**])

$$ko_{2+w}(B\mathrm{O}_1) \cong \mathbb{Z}/8$$

and thus the class $w_{1!}([\Sigma])$ may be interpreted as an 8th root of unity via the exponential map.

Proposition 3.2.38. *The assignment $M \mapsto w_{1!}([M])$ defines an isomorphism*

$$\Omega_2^{\mathrm{Pin}^-} \xrightarrow{\sim} ko_{2+w}(B\mathrm{O}_1)$$

and thus after a suitable choice of isomorphism $ko_{2+w}(B\mathrm{O}_1) \simeq \mu_8$, we may identify $w_{1!}([M])$ with the Arf-Brown invariant.

Remark 3.2.39. The advantage of this perspective is that it naturally occurs as the induced map on homotopy groups of a map of spectra. Namely, the observation from §3.2.3.2 relating pin$^-$ structures and spin structures leads to the identification

$$MTPin^- \simeq MPin^+ \simeq MSpin \wedge \Sigma^{-1}MO_1. \tag{3.2.40}$$

Thus smashing the Atiyah-Bott-Shapiro orientation with the factor $\Sigma^{-1}MO_1$ leads to a map of spectra

$$MTPin^- \to ko \wedge \Sigma^{-1}MO_1. \tag{3.2.41}$$

The class $w_{1!}([M])$ as defined in (3.2.37) is precisely the image of the class in $\Omega_2^{\mathrm{Pin}^-}$ defined by M.

Proof of Proposition 3.2.38. The morphism $\Omega_2^{\mathrm{Pin}^-} \to ko_{2+w}(B\mathrm{O}_1)$ given by $M \mapsto w_{1!}([M])$ is just the induced morphism on the homotopy groups π_2 associated to the map of spectra (3.2.41). To check that we get an isomorphism on π_2, note that the original Atiyah-Bott-Shapiro map (3.2.40) is 8-connected (see [**ABP67**]) and thus it remains so after taking the smash product with the connective spectrum $\Sigma^{-1}MO_1$, in particular inducing an isomorphism on π_2 as required. $\square$

Remark 3.2.42. One may define an invariant in $\mathbb{Z}/8$ associated to any vector bundle on Σ by first considering the twisted Poincaré duality isomorphism

$$KO^0(\Sigma) \to KO_{2+w_1}(\Sigma),$$

then taking the pushforward in twisted KO-homology as before.

We now mention a few closely related KO-theoretic constructions of the Arf-Brown invariant that appear in the literature.

3.2.3.4. *Zhang's construction.* Given a pin$^-$ surface Σ, the classifying map of the orientation double cover

$$w_1 : \Sigma \to B\mathrm{O}_1 \simeq \mathbb{RP}^\infty$$

is homotopic to a map which factors through the 2-skeleton $\mathbb{RP}^2$. Let $\widetilde{w_1}$ denote this map $\Sigma \to \mathbb{RP}^2$. The stable normal bundle of $\widetilde{w_1}$ is a virtual vector bundle of rank 0 which carries a spin structure (after choosing a pin$^-$ structure on $\mathbb{RP}^2$), and thus there is a well-defined pushforward map

$$KO^*(\Sigma) \to KO^*(\mathbb{RP}^2).$$

In [**Zha17**], Zhang defines the topological index of a vector bundle V on Σ as the image of $f_! V$ under a certain homomorphism

$$KO^0(\mathbb{RP}^2) \cong \mathbb{Z} \oplus \mathbb{Z}/4 \to \mathbb{Z}[1/4]/2 \cong \mathbb{Z}/8.$$

It is shown in loc. cit. that the topological index agrees with the reduced η-invariant of a twisted Dirac operator on V. In particular, the topological index of the trivial bundle agrees with the Arf-Brown invariant.

One can check that Zhang's construction of the topological index agrees with the one in Remark 3.2.42 (first observe that in each case, the index factors through $KO^0(\mathbb{RP}^2)$, then check that the morphism to $\mathbb{Z}/8$ is the same).

3.2.3.5. *Distler-Freed-Moore construction.* Distler-Freed-Moore [**DFM10**] give a slightly different KO-theoretic construction. In order to state it, we will need to consider two modifications of the cohomology theory KO: first, we consider the Postnikov truncation of the connective cover $R \coloneqq \tau_{0:4}KO$; then we take R-cohomology with coefficients in $\mathbb{R}/\mathbb{Z}$ (this is represented by a spectrum $R(\mathbb{R}/\mathbb{Z})$ which belongs to a fiber sequence $R \to R \wedge H\mathbb{R} \to R(\mathbb{R}/\mathbb{Z})$). Moreover, the authors consider a twisted version of this theory, as explained above.

It is shown in [**DFM10**] that $R^{w_1-2}(BO_1; \mathbb{R}/\mathbb{Z})$ is cyclic of order 8. Given a closed pin$^-$ surface Σ, the Arf-Brown invariant is obtained as the image of a generator under the morphisms

$$R^{w_1-2}(BO_1; \mathbb{R}/\mathbb{Z}) \xrightarrow{w_1^*} R^{w_1-2}(\Sigma; \mathbb{R}/\mathbb{Z}) \xrightarrow{\pi_!^\Sigma} R^{-4}(\mathrm{pt}; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} \xrightarrow{\exp} \mathbb{C}^\times.$$

3.3. Invertible TFTs via stable homotopy theory

In this section, we give a brief exposition of TFTs leading to the classification of invertible TFTs in terms of homotopy theory. Some of this is a review from §1.2 and §1.3, but since we care only about two-dimensional theories in this chapter, we will focus the exposition on them, which changes the exposition slightly.

3.3.1. What is a (2d) invertible TFT?.

3.3.1.1. *Atiyah-Segal Axioms.* In [**Ati88**] (see also [**Seg88**]) an n-dimensional TFT is axiomatized as a symmetric monoidal functor

$$Z : \mathsf{Bord}_{n,n-1} \to \mathsf{Vect}_{\mathbb{C}}.$$

The source category has objects given by closed $(n-1)$-manifolds, and morphisms are bordisms between them. Thus a TFT Z will assign a vector space $Z(Y^{n-1})$ to a closed $(n-1)$-manifold, and a linear map

$$Z(X) : Z(Y_1) \to Z(Y_2)$$

whenever X is a bordism between Y_1 and Y_2. Moreover, this assignment is compatible with the symmetric monoidal structures on each side: disjoint union of manifolds is taken to tensor product of vector spaces. In particular, the empty $(n-1)$-manifold $\varnothing^{n-1}$ is the unit object for the symmetric monoidal structure on $\mathsf{Bord}_{n,n-1}$, and thus we can identify $Z(\varnothing^{n-1})$ with the trivial line $\mathbb{C}$. Given a closed n-manifold, X^n, we may interpret it as a bordism between empty $(n-1)$-manifolds, and thus we obtain a number $Z(X) \in \mathbb{C}$; this is referred to as the *partition function* of the theory.

In order to study the Arf-Brown TFT, we will need a number of variations, extensions, and simplifications of these axioms.

3.3.1.2. *Fully extended TFTs.* A 2-dimensional TFT as defined above may be thought of as an assignment of an invariant $Z(X) \in \mathbb{C}$ to every closed 2-manifold X, which satisfies a certain locality property. Namely,

$Z(X)$ may be computed by decomposing X in to pairs of pants and discs. We will be interested in *fully extended TFTs* which satisfy a stronger form of locality, allowing the partition function to be computed by decomposing X in to arbitrarily small pieces (for example, a triangulation).

One way to make this precise involves replacing the ordinary category $\mathsf{Bord}_{2,1}$ with a certain 2-category Bord_2. In this paper, we will use the term 2-category to denote what some authors would call a *weak 2-category*, or *bicategory*. We will treat the subject in an informal and expository manner—a 2-category should have objects, 1-morphisms, and 2-morphisms which may be composed in various ways; for further details, see [**Bén67, SP17**]. In our case, the objects of Bord_2 are now closed 0-manifolds, 1-morphisms are 1-dimensional bordisms, and 2-morphisms are diffeomorphism classes of certain 2-manifolds with corners, interpreted as bordisms between 1-dimensional bordisms.

Similarly we must replace the category $\mathsf{Vect}_{\mathbb{C}}$ with an appropriate 2-category of coefficients for the theory. One choice is the Morita 2-category $\mathsf{Alg}_{\mathbb{C}}$, whose objects are algebras, 1-morphisms are bimodules, and 2-morphisms are bimodule homomorphisms. As in the usual Atiyah-Segal axioms, these 2-categories carry symmetric monoidal structures (for more details on symmetric monoidal 2-categories, see [**Shu10**]). A fully extended 2d TFT may be formulated as a symmetric monoidal functor

$$Z : \mathsf{Bord}_2 \to \mathsf{Alg}_{\mathbb{C}}.$$

Note that the original categories $\mathsf{Bord}_{1,2}$ and $\mathsf{Vect}_{\mathbb{C}}$ sit as the endomorphism categories of the unit objects in Bord_2 and $\mathsf{Alg}_{\mathbb{C}}$ respectively, and thus a fully extended TFT gives rise a TFT as in Atiyah and Segal's original definition.

3.3.1.3. *Tangential structure.* As the Arf-Brown invariant is an invariant of pin$^-$ surfaces, to define the Arf-Brown TFT, we must consider a variant of the bordism category in which all the manifolds are equipped with a pin$^-$ structure. More generally, for any Lie group H equipped with a homomorphism $H \to O_2$, there is a 2-category Bord_2^H defined as above, but now all manifolds are equipped with H-structures.[3]

Remark 3.3.1 (Structure groups)**.** In the cases of interest, the group H is usually part of a family H_n, $n \in \mathbb{Z}_{>0}$, equipped with maps $H_n \to \mathrm{O}_n$. For example, we could take H_n to be SO_n, Spin_n, $\mathrm{Pin}_n^{\pm}$, or O_n itself. These examples all have the additional property that an H_n-structure on a vector bundle V is equivalent to an H_{n+1}-structure on $V \oplus \underline{\mathbb{R}}$; thus we may think of an H_n structure as an H-structure on the corresponding stable vector bundle (where $H = \varinjlim H_n$). In that case, we will generally write Bord_2^H instead of $\mathsf{Bord}_2^{H_2}$. We

[3]One defines an H-structure on a 0- or 1-manifold by taking the direct sum of the tangent bundle with a trivial bundle of appropriate rank (so the total rank is two). Or better, one can consider the manifolds appearing in the bordism category as being equipped with 2-dimensional collars. Making this precise in the smooth category takes some care; see [**SP17**] for a careful treatment in the 2-dimensional case.

will also consider the framed case $H_n = 1$; however, note that a framing on an n-dimensional bundle is not determined by a stable framing in general.

Remark 3.3.2. Given a symmetric monoidal 2-category C, we say that C *has duals* if every object of C admits a dual with respect to the monoidal structure and every 1-morphism of C admits an adjoint. The bordism 2-categories $\mathsf{Bord}_2^{H_2}$ all have duals in this sense. Moreover, one version of the cobordism hypothesis [**Lur09b**] states that the framed bordism category Bord_2^{fr} (obtained by taking $H_2 = 1$ above) is universal for this property, in the following sense: given any symmetric monoidal 2-category with duals C, symmetric monoidal functors $\mathsf{Bord}_2^{fr} \to \mathsf{C}$ are in correspondence with objects of C (where the correspondence assigns to a functor, the value of the framed 0-manifold pt_+).

3.3.1.4. *Gradings.* To allow for more interesting theories we can also enlarge the coefficient category by considering algebras and bimodules with a $\mathbb{Z}/2$-grading (and morphisms compatible with this grading). This gives rise to a 2-category $\mathsf{sAlg}_{\mathbb{C}}$ (the 2-category of superalgebras), which is equipped with a symmetric monoidal structure incorporating the Koszul rule of signs. Later we will see that this choice of target category is universal in a certain sense (see Corollary 3.3.23).

3.3.1.5. *Examples.*

Example 3.3.3 (Euler Theories)**.** There is a TFT Z_1 defined on unoriented manifolds which assigns only identity objects and morphisms in $\mathsf{sAlg}_{\mathbb{C}}$. In particular, the partition function takes the constant value 1. Slightly more interesting is the *Euler theory* Z_λ, associated to a complex number $\lambda \in \mathbb{C}^\times$. This agrees with the trivial theory on 0 and 1-manifolds, but assigns the number $\lambda^{\chi(X)}$ (considered as a linear map between trivial lines) to any 2-dimensional bordism.

Example 3.3.4 (2-dimensional Dijkgraaf-Witten Theory [**DW90, FQ93, FHLT10**])**.** Given a finite group G, there is a 2d oriented TFT whose partition function on a closed surface is the weighted sum

$$Z_G(\Sigma) = \sum_{[P]} \frac{1}{|\operatorname{Aut}(P)|}.$$

This theory assigns the group algebra of G to a point, and the space of class functions to a circle.

Example 3.3.5 (The Arf-Brown Theory)**.** We will see in 3.4.0.1 that the Arf-Brown invariant is the partition function of a 2d fully extended TFT. In other words, there is a symmetric monoidal functor

$$Z_{AB} : \mathsf{Bord}_2^{\mathrm{Pin}^-} \to \mathsf{sAlg}_{\mathbb{C}}$$

such that for a pin$^-$ surface X, $Z_{AB}(X) = AB(X)$. The Arf-Brown theory takes the following values on lower dimensional manifolds:

- For a bounding pin$^-$ circle S^1_b, $Z_{AB}(S^1_b) \cong \mathbb{C}$, an even line.
- For a non-bounding pin$^-$ circle S^1_{nb}, $Z_{AB}(S^1_{nb}) \cong \mathbb{C}[1]$, an odd line.
- We have $Z_{AB}(\mathrm{pt}) \cong C\ell_1$, the first Clifford algebra.

3.3.1.6. *Invertible theories.* The Euler theories and the Arf-Brown theory have the property that every value of the functor Z is invertible (either as an object with respect to the monoidal structure, or as a 1- or 2-morphism) in the symmetric monoidal 2-category $\mathsf{sAlg}_{\mathbb{C}}$ (for example, Dijkgraaf-Witten theory does not have this property unless $G = 1$). Such TFTs are called *invertible.*

When dealing with invertible theories we may restrict attention to the following class of 2-categories:

Definition 3.3.6. A 2-category is called a 2-*groupoid* if every 1-morphism and 2-morphism is invertible. A symmetric monoidal 2-category C is called a *Picard 2-groupoid* if it is a 2-groupoid and in addition every object is invertible with respect to the monoidal structure.

Note that if a symmetric monoidal 2-groupoid C has duals, then it is necessarily a Picard 2-groupoid, i.e. the duals of objects must be inverses.

Given a symmetric monoidal 2-category C, we may consider the maximal subcategory $\mathsf{C}^\times$ which is a Picard 2-groupoid (i.e. throw out any non-invertible objects and morphisms). A TFT

$$Z : \mathsf{Bord}_2^{H_2} \to \mathsf{sAlg}_{\mathbb{C}}$$

is invertible if and only if Z factors through $\mathsf{sAlg}_{\mathbb{C}}^\times \to \mathsf{sAlg}_{\mathbb{C}}$.

There is another way to associate a 2-groupoid $\overline{\mathsf{C}}$ to a 2-category C by formally inverting any non-invertible 1 and 2-morphisms. This procedure is left adjoint to the inclusion of 2-groupoids in to 2-categories (the maximal 2-groupoid is right adjoint). In other words, any functor from C to a 2-groupoid will factor uniquely through $\overline{\mathsf{C}}$. Moreover, if C is symmetric monoidal and has duals, then $\overline{\mathsf{C}}$ will be a Picard 2-groupoid. We note the following upshot: the data of an invertible (fully extended) H_2-TFT is given by a functor of Picard 2-groupoids

$$Z\colon \overline{\mathsf{Bord}_2^{H_2}} \longrightarrow \mathsf{sAlg}_{\mathbb{C}}^\times.$$

3.3.2. The homotopy hypothesis and stable 2-types. Now let us explain how an invertible TFT may be reformulated in terms of maps in the stable homotopy category.

3.3.2.1. *The fundamental 2-groupoid and Postnikov truncations.* To begin with let us recall the following 2-category associated to a space.

Example 3.3.7. Let X be a topological space. The *fundamental 2-groupoid* $\pi_{\leq 2}(X)$ of X is given as follows:

- the objects are points of X,
- the 1-morphisms are paths between points, and
- the 2-morphisms are given by homotopy classes of homotopies between paths.

To understand what information about a space the fundamental 2-groupoid captures, let us recall the following:

Definition 3.3.8. We say that a topological space X is a *homotopy n-type* if the homotopy groups $\pi_i(X)$ are non-zero only if $i = 0, 1, \dots n$. Given any space X, there is a Postnikov fibration

$$f_{\leq n} : X \to X_{\leq n}$$

where $X_{\leq n}$ is an n-type and $f_{\leq n}$ induces an isomorphism on π_i for $i = 0, 1, \dots n$. We refer to $X_{\leq n}$ as the *(Postnikov) n-truncation* of X, or simply the *n-type* of X.

3.3.2.2. *Unstable homotopy hypothesis.* The idea of the homotopy hypothesis (formulated by Grothendieck in "Pursuing Stacks") is that the homotopy theory of n-types should be modeled by n-groupoids. There are many forms the homotopy hypothesis might take, depending on what flavor of higher category theory one considers. In some cases, the result is almost tautological (if ones uses an inherently homotopical theory of higher category), and in others it is false (if one uses a too strict a higher category theory).

Here's one form of the homotopy hypothesis in the case $n = 2$.[4]

Theorem 3.3.9 (2-dimensional homotopy hypothesis [**Noo07**, Prop. 9.8])**.** *The assignment from a homotopy 2-type to its fundamental 2-groupoid defines an equivalence of categories between the homotopy category of 2-types and the category of 2-groupoids, with morphisms equivalence classes of 2-functors.*

The inverse functor is given by the *classifying space* of a 2-groupoid. In general, the classifying space of a 2-category assigns a space $|\mathsf{C}|$ to a 2-category C (see [**Dus02**]), constructed as the geometric realization of a certain simplicial set—the *nerve* of C.

[4]We thank the referee for finding a reference for this theorem.

Remark 3.3.10. The classifying space of a 2-category C is weakly equivalent to the classifying space of its localization $\overline{\mathsf{C}}$ (in fact, one may construct the localization by taking the fundamental 2-groupoid of its classifying space).

3.3.2.3. *Stable homotopy hypothesis.* A symmetric monoidal structure on a 2-category C induces a binary operation on the classifying space $|\mathsf{C}|$. This operation is commutative and associative up to homotopy; more precisely, it can be shown that $|\mathsf{C}|$ carries an E_∞ structure. If C has duals (for example, if it is a Picard 2-groupoid), then every object has an inverse up to homotopy, and the E_∞ structure is said to be grouplike.

Foundational results in homotopy theory (see [**Ada78**, §2.3] and the references therein, or [**Lur17**, §5.1.3] for a modern approach) state that a grouplike E_∞-structure on a space X is equivalent to an infinite loop space structure on X. Equivalently, $X = \Omega^\infty E$ may be identified as the zeroth space in a connective spectrum E (the other spaces in the spectrum are given by iterated deloopings or loop spaces of X).

Definition 3.3.11. A *stable n-type* is a connective spectrum E such that $\Omega^\infty E$ is a homotopy n-type.

Thus we arrive at the *stable homotopy hypothesis*, which states that there is an equivalence between:

- The homotopy category stable 2-types.
- The category of Picard 2-groupoids with equivalence classes of symmetric monoidal functors.

Thanks to recent work of [**GJO19**] this is now a precisely formulated theorem.

3.3.3. Classifying invertible TFTs up to isomorphism. One consequence of the stable homotopy hypothesis is that we may encode a 2d H_2-TFT as a morphism of stable 2-types:

$$|Z|\colon |\overline{\mathsf{Bord}_2^{H_2}}| \to |\mathsf{sAlg}_{\mathbb{C}}^{\times}|.$$

Thus to classify isomorphism classes of invertible TFTs, we should classify homotopy classes of maps between stable 2-types as above.

3.3.3.1. *Stable Postnikov data.* Let us first unpack the case of a stable 1-type, which according to the appropriate version of the homotopy hypothesis, is equivalent data to a Picard groupoid [**JO12**, Theorem 1.5].

A stable 1-type E may be succinctly encoded in terms of the following data:

- a pair of abelian groups $A = \pi_0 E$ and $B = \pi_1 E$, and
- a homomorphism $k : A/2 \to B$.

The homomorphism k encodes the k-invariant

$$HA \to \Sigma^2 HB$$

which defines the Postnikov tower of E.

Remark 3.3.12 ([**GJOS17**, §3])**.** The homomorphism k may also be identified with the action of the generator $\eta \in \pi_1(\mathbb{S}) \cong \mathbb{Z}/2$ on the homotopy groups of E (see Remark 3.3.20).

Proposition 3.3.13 ([**JO12**])**.** *Given a Picard 1-groupoid C, we recover the above data as follows:*

- *$\pi_0(|\mathsf{C}|)$ is the set of equivalence classes of objects in C, with group structure coming from the symmetric monoidal structure.*
- *$\pi_1(|\mathsf{C}|)$ is the set of automorphisms of the unit object in C.*
- *Given an object $a \in \mathsf{C}$, the element $k(a) \in \pi_1(|\mathsf{C}|)$ may be identified with the image of the symmetry map $\sigma \in \mathrm{Aut}(a \otimes a)$ under the equivalence $\mathrm{Aut}(1_{\mathsf{C}}) \simeq \mathrm{Aut}(a \otimes a)$ induced by the functor $- \otimes (a \otimes a)$.*

Similarly, a stable 2-type may be described by the following data:

(3.3.14)
$$\begin{tikzcd}
\Sigma^2 HC \arrow[r, "i_2"] & E = E\langle 0,1,2\rangle \arrow[d] & \\
\Sigma^1 HB \arrow[r, "i_1"] & E\langle 0,1\rangle \arrow[r, "k_1"] \arrow[d] & \Sigma^3 HC \\
\Sigma HA \arrow[r, "i_0"] & E\langle 0\rangle \arrow[r, "k_0"] & \Sigma^2 HB
\end{tikzcd}$$

As we have seen considering stable 1-types, the maps $k_0 i_0$ and $k_1 i_1$ are classified by homomorphisms

(3.3.15)
$$A/2 \to B$$

and

(3.3.16)
$$B/2 \to C.$$

The map i_0 is an equivalence, so k_0 is determined by $k_0 i_0$. However, these data do not fix the homotopy class of k_1 itself and thus do not fix the homotopy type of E in general.[5]

[5]For example, consider the Postnikov truncation $\tau_{\leq 2} ku$ of connective complex K-theory. We have that $\pi_1(ku) = 0$, so both (3.3.15) and (3.3.16) are necessarily zero; however, there is a non-zero k-invariant $H\mathbb{Z} \to \Sigma^3 H\mathbb{Z}$.

3.3.3.2. *Bordism and Madsen-Tillmann Spectra.* Given a Lie group H_n with a map $H_n \to \mathrm{O}_n$, Galatius-Madsen-Tillmann-Weiss [**GMTW09**] define the Madsen-Tillmann spectrum MTH_n as the Thom spectrum of the virtual vector bundle $-\gamma_n$ over BH_n (where γ_n is pulled back from the tautological vector bundle on $B\mathrm{O}_n$). Building on the fundamental work of [**GMTW09, MW07**], Schommer-Pries [**SP17**] shows that the classifying space of $\mathsf{Bord}_2^{H_2}$ is given by the Postnikov 2-truncation of $\Omega^{\infty-2}MTH_2$.[6]

Let us also consider the following variant of the bordism 2-category. Now suppose H_n is a family of groups as in Remark 3.3.1. Associated to such data we have the *stable bordism 2-category*, a symmetric monoidal 2-category $\mathsf{Bord}_{2,st}^{H}$ defined in a similar way to $\mathsf{Bord}_2^{H_2}$, except the manifolds are now equipped with a H-structure on their stabilized tangent bundles (the direct limit of the sequence obtained by taking iterated direct sums with the trivial vector bundle), and crucially that 2-morphisms are given by 2-bordisms modulo the relation given by 3-bordisms (as opposed to diffeomorphism as before).

Remarkably, the stable bordism 2-category is already a Picard 2-groupoid (the duals in the usual bordism 2-category are inverses modulo bordism). Its classifying space is the stable 2-type associated to the spectrum MTH, the direct limit of spectra $\Sigma^n MTH_n$, which represents the cohomology theory defined by H-bordism of manifolds: $\pi_i(MTH) = \Omega_i^H$ consists of smooth closed i-manifolds, with a H-structure on the stable tangent bundle, up to H-bordism.

Example 3.3.17. In the case $H = 1$, we have the *framed* stable 2d bordism category, $\mathsf{Bord}_{2,st}^{fr}$. Pontrjagin-Thom theory identifies the classifying space of this category with the stable 2-type of the sphere spectrum $\mathbb{S}$. The homotopy groups are given by

$$\pi_0(\mathbb{S}) \cong \mathbb{Z}, \pi_1(\mathbb{S}) \cong \mathbb{Z}/2, \pi_2(\mathbb{S}) \cong \mathbb{Z}/2.$$

The generator η of $\pi_1(\mathbb{S})$ is represented by the stably framed manifold S^1 with its Lie group framing, and the generator η^2 of $\pi_2(\mathbb{S})$ is represented by $S^1 \times S^1$, also with its Lie group framing (see Remark 3.3.20).

Let H_n be a family of groups as in Remark 3.3.1. There is a canonical functor

$$\mathsf{Bord}_2^{H_2} \to \mathsf{Bord}_{2,st}^{H}.$$

Definition 3.3.18. An H_2-TFT

$$Z : \mathsf{Bord}_2^{H_2} \to \mathsf{sAlg}_{\mathbb{C}}$$

is called *stable* if it factors through $\mathsf{Bord}_{2,st}^{H}$.

[6]More generally, in [**SP17**] it is shown that the classifying space of the (∞, n)-category $\mathsf{Bord}_n^{H_n}$ is $\Sigma^n MTH_n$.

Example 3.3.19. We have described the homotopy groups of

$$|\overline{\mathsf{Bord}_{2,st}^{\mathrm{Pin}^-}}| \simeq \tau_{0:2} MTPin^-$$

in Proposition 3.1.16; in particular, recall that $\pi_2(MTPin^-) = \Omega_2^{\mathrm{Pin}^-} \simeq \mathbb{Z}/8$. By the results of [**GMTW09, Ngu17, SP17**], there is an equivalence:

$$|\overline{\mathsf{Bord}_2^{\mathrm{Pin}^-}}| \simeq \tau_{0:2}\big(\Sigma^2 MTPin_2^-\big).$$

According to [**RW14**], we have $\pi_2(\Sigma^2 MTPin_2^-) \cong \mathbb{Z} \oplus \mathbb{Z}/4$, and the map

$$\mathsf{Bord}_2^{\mathrm{Pin}^-} \to \mathsf{Bord}_{2,st}^{\mathrm{Pin}^-}$$

induces the map $\mathbb{Z} \oplus \mathbb{Z}/4 \to \mathbb{Z}/8$ which takes (a, b) to $a + 2b \mod 8$. The projection on to the first factor is represented by the Euler characteristic. In particular, we see that the Euler theory Z_λ of Example 3.3.3 is stable if and only if $\lambda = \pm 1$.

Remark 3.3.20. The Thom spectra MO, $MSpin$, and $\mathbb{S}$ carry natural E_∞-ring structures; geometrically, the ring operations corresponds to direct product of manifolds. Moreover, any spectrum is a module spectrum for the sphere spectrum in a unique way (just as any any abelian group is a module for the integers). In particular, the graded ring $\bigoplus_i \pi_i(\mathbb{S})$ acts on $\bigoplus_i \pi_i(E)$ for any spectrum E. In the case E is a bordism spectrum then this action may be understood in terms of direct products of appropriately structured manifolds. By Lemma 1.1.17, though, $MPin^\pm$ are not ring spectra – only module spectra for $MSpin$.

3.3.3.3. *Brown-Comenetz duality and invertible superalgebras.* Recall from Definition 1.1.47 the definition of the Brown-Comenetz dual to the sphere spectrum $I_{\mathbb{C}^\times}$, which satisfies the property that $[X, \Sigma^n I_{\mathbb{C}^\times}] \cong \mathrm{Hom}(\pi_n(X), \mathbb{C}^\times)$. In particular, $I_{\mathbb{C}^\times}$ is coconnective — its positive-degree homotopy groups vanish. In particular, a character $c_k \in \pi_k(E)^\vee$ determines morphisms for each positive integer i:

$$c_{k-i} : \pi_{k-i}(E) \to \pi_{k-i}(\Sigma^k I\mathbb{C}^\times) = \pi_i(\mathbb{S})^\vee.$$

Unwinding the definitions, we see that these morphisms are computed by the action of $\pi_*(\mathbb{S})$ on $\pi_*(E)$:

Lemma 3.3.21. *Given E and $c_k \in \pi_k(E)^\vee$ as above, we have*

$$c_{k-i}(x)(\xi) = c_k(\xi \cdot x)$$

for all $x \in \pi_{k-i}(E)$, $\xi \in \pi_i(\mathbb{S})$.

The following key result identifies the Picard 2-groupoid corresponding to $\Sigma^2 I_{\mathbb{C}}^{\times}$.

Proposition 3.3.22. *The stable* 2*-type* $|\mathsf{sAlg}_{\mathbb{C}}^{\times}|$ *is equivalent to the connective cover of* $\Sigma^2 I_{\mathbb{C}}^{\times}$*.*

We give a proof of this result in 3.3.3.5 below.

As an immediate consequence of Proposition 3.3.22 we obtain the following notable result:

Corollary 3.3.23. *Any character* $c\colon \pi_2(MTH_2) \to \mathbb{C}^{\times}$ *arises as the partition function of a unique invertible TFT*

$$Z_c\colon \mathsf{Bord}_2^{H_2} \to \mathsf{sAlg}_{\mathbb{C}}.$$

Example 3.3.24. Continuing with Example 3.3.19, we see that isomorphism classes of invertible pin$^-$ TFTs are given by

$$(\mathbb{Z} \oplus \mathbb{Z}/4)^{\vee} = \mathbb{C}^{\times} \times \mu_4.$$

On the other hand, the stable theories are represented by the cyclic subgroup generated by (ζ_8, ζ_4) (where ζ_4, resp. ζ_8, denotes a primitive 4th, resp. 8th, root of unity).

3.3.3.4. *Deformation classes of theories and the Freed-Hopkins classification.* The natural computation from the perspective of topological phases is not to do with *isomorphism classes* of theories, but rather *deformation classes*. Informally this may be understood in terms of replacing the 2-category $\mathsf{sAlg}_{\mathbb{C}}$ with an appropriate topological 2-category in which $\mathbb{C}$ is considered with its continuous topology. More precisely, one considers the *Anderson dual* spectrum $I_{\mathbb{Z}}$ which may be defined as the homotopy fiber of the map $H\mathbb{C} \to I_{\mathbb{C}}^{\times}$ given by the exponential map. In particular, there is a map

$$|\mathsf{sAlg}_{\mathbb{C}}^{\times}| \simeq \Sigma^2 I_{\mathbb{C}}^{\times} \to \Sigma^3 I_{\mathbb{Z}},$$

which can be understood as passing from the discrete to the continuous topology on $\mathbb{C}$. Deformation classes of 2-dimensional invertible theories may then be computed in terms of homotopy classes of maps in to $\Sigma^3 I_{\mathbb{Z}}$.

In fact, the relevant computation for the classification of symmetry protected phases concerns the deformation classes of reflection positive theories. Freed-Hopkins define an invertible TFT with *reflection structure* to be one whose associated map of spectra is equivariant with respect to certain involutions on the domain and codomain [**FH16a**, Ansatz 7.8], and a TFT with *reflection positivity structure* to be one with reflection structure and a trivialization of an associated map [**FH16a**, Definition 8.20]. Reflection structure implies that *isomorphism* classes of reflection positive invertible TFTs are classified by homotopy classes of $\mathbb{Z}/2$-equivariant maps of Borel $\mathbb{Z}/2$-equivariant spectra, but Freed-Hopkins prove [**FH16a**, Theorem 8.23] that *deformation* classes of reflection positive invertible TFTs can be classified in terms of nonequivariant maps of spectra, and that up to deformation, a reflection positive theory can always be represented by a stable theory. In the case of interest in this paper (2-dimensional pin$^-$ theories) deformation classes of reflection positive theories are in natural bijection with isomorphism classes of stable theories: both groups are cyclic of order 8.

3.3.3.5. *Proof of 3.3.22.* By the defining property of $I_{\mathbb{C}}^{\times}$, there is a unique homotopy class of maps of spectra

$$c : |\mathsf{sAlg}_{\mathbb{C}}^{\times}| \to \Sigma^2 I_{\mathbb{C}}^{\times}$$

which induces the identity map

$$c_2 = \pi_2(c) : \mathbb{C}^{\times} = \pi_2(|\mathsf{sAlg}_{\mathbb{C}}^{\times}|) \to \pi_2(\Sigma^2 I\mathbb{C}^{\times}) = \pi_0(\mathbb{S})^{\vee} = \mathbb{C}^{\times}.$$

To prove the proposition, we must check that the morphism c_2 induces isomorphisms

$$c_1 : \mathbb{Z}/2 \cong \pi_1(|\mathsf{sAlg}_{\mathbb{C}}^{\times}|) \to \pi_1(\Sigma^2 I_{\mathbb{C}}^{\times}) = \pi_1(\mathbb{S})^{\vee} \cong \mu_2$$

and

$$c_0 : \mathbb{Z}/2 \cong \pi_0(|\mathsf{sAlg}_{\mathbb{C}}^{\times}|) \to \pi_0(\Sigma^2 I_{\mathbb{C}}^{\times}) = \pi_2(\mathbb{S})^{\vee} \cong \mu_2.$$

By Lemma 3.3.21, to understand the maps c_0 and c_1, we must compute the action of $\pi_1(\mathbb{S})$ and $\pi_2(\mathbb{S})$ on $\pi_*(|\mathsf{sAlg}_{\mathbb{C}}^{\times}|)$.

First consider the generator $\eta \in \pi_1(\mathbb{S}) \cong \mathbb{Z}/2$. Recall from Remark 3.3.12 that the action of η on the homotopy groups of the classifying space of a Picard groupoid is given by the formula in Proposition 3.3.13 (which also encodes the unique k-invariant of the stable 1-type). Consider the Picard groupoid $\mathsf{sVect}_{\mathbb{C}}^{\times}$, whose classifying space is weakly equivalent to $\Omega|\mathsf{sAlg}_{\mathbb{C}}^{\times}|$. As explained in Proposition 3.3.13, the action of η is given by the homomorphism

$$\pi_1(|\mathsf{sAlg}_{\mathbb{C}}^{\times}|) \otimes \mathbb{Z}/2 = \pi_0(|\mathsf{sVect}_{\mathbb{C}}^{\times}|) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2 \to \pi_1(|\mathsf{sVect}_{\mathbb{C}}^{\times}|) = \mathbb{C}^{\times}$$

which takes the generator (represented by an odd line) to the number $-1 \in \mathbb{C}^{\times}$. It follows that c_1 applied to the class of on odd line gives the unique non-trivial character of $\pi_1(\mathbb{S}) \cong \mathbb{Z}/2$, and thus c_1 is an isomorphism as required.

Now $\pi_2(\mathbb{S}) \cong \mathbb{Z}/2$ is generated by η^2. Thus, the action of η^2 is a composite:

$$\pi_0(|\mathsf{sAlg}_{\mathbb{C}}^{\times}|) \xrightarrow{\eta} \pi_1(|\mathsf{sAlg}_{\mathbb{C}}^{\times}|) \xrightarrow{\eta} \pi_2(|\mathsf{sAlg}_{\mathbb{C}}^{\times}|).$$

Thus it remains to compute the map between π_0 and π_1. For this, we consider the Picard groupoid representing the $\langle 0,1\rangle$ Postnikov truncation of $|\mathsf{sAlg}_{\mathbb{C}}^{\times}|$; its objects are Morita invertible complex superalgebras, and morphisms are *isomorphism classes* of invertible bimodules. The action of η is computed via the same method as before using the symmetry isomorphism. One sees that the generator of $\pi_0(|\mathsf{sAlg}_{\mathbb{C}}^{\times}|)$ (represented by the Clifford algebra $C\ell_1$) is taken by η to the non-trivial class $\pi_1(|\mathsf{sAlg}_{\mathbb{C}}^{\times}|)$ represented by the odd line.

Putting everything together we observe that c_0 takes the class of $C\ell_1$ to the unique non-trivial character of $\pi_2(\mathbb{S})$, and thus is an isomorphism as required.

3.4. The Arf-Brown TFT

3.4.0.1. *The Arf-Brown Theory.* In particular, recall the Arf-Brown invariant

$$AB : \Omega_2^{\mathrm{Pin}^-} \to \mu_8 \subseteq \mathbb{C}^{\times}.$$

We denote by

$$Z_{AB} : \mathsf{Bord}_2^{\mathrm{Pin}^-} \to \mathsf{sAlg}_{\mathbb{C}}$$

the unique TFT with partition function given by AB (see Corollary 3.3.23).

Proposition 3.4.1. *The pin$^-$ Arf-Brown TFT*

$$Z_{AB} : \mathsf{Bord}_2^{\mathrm{Pin}^-} \to \mathsf{sAlg}_{\mathbb{C}}$$

assigns the following invariants:

- *To a pin$^-$ point, Z_{AB} assigns the first Clifford algebra $C\ell_1$.*
- *To a bounding pin$^-$ circle, Z_{AB} assigns an even line $\mathbb{C}$.*
- *To a non-bounding pin$^-$ circle, Z_{AB} assigns an odd line $\mathbb{C}[1]$.*

Proof. We use Lemma 3.3.21 to compute the values of Z_{AB} on closed manifolds in terms of the action of $\pi_*(\mathbb{S})$ on $\pi_*(MTPin^-)$.

Recall that the homotopy groups of $\mathbb{S}$ (respectively $MTPin^-$) are given by bordism classes of stably framed manifolds (respectively, pin$^-$-manifolds). As usual, let η denote the generator of $\pi_1(\mathbb{S})$, which is represented by the bordism class of the circle S^1 with its Lie group framing (which induces the non-bounding pin$^-$ structure). Thus the operation "multiplication by η" on $\pi_*(MTPin^-)$ may be understood as direct product with the pin$^-$ manifold S^1_{nb}.

We have [**KT90b**] that the class of S^1_{nb} is a generator of $\pi_1(MTPin^-) \cong \mathbb{Z}/2$. The class of $S^1_{nb} \times S^1_{nb}$ is the unique element of order 2 in $\pi_2(MTPin^-)$, and its Arf-Brown invariant is $-1 \in \mathbb{C}^{\times}$.

It follows that Z_{AB} takes the non-bounding circle to the unique non-trivial character of $\pi_1(\mathbb{S})$ (which takes the class η represented by S^1_{nb} to -1). Similarly, Z_{AB} takes the pin$^-$ point to the unique non-trivial character of $\pi_2(\mathbb{S})$ (which takes the class η^2 represented by $S^1_{nb} \times S^1_{nb}$ to -1), as required. $\square$

For a different construction of the nonextended Arf-Brown TFT, see [**Yon19**, §4.2].

Remark 3.4.2. The Arf-Brown theory gives rise to an invertible spin TFT as the composite

$$Z_A : \mathsf{Bord}_2^{\mathrm{Spin}} \to \mathsf{Bord}_2^{\mathrm{Pin}^-} \to \mathsf{sAlg}_{\mathbb{C}}$$

which assigns the Arf invariant to a closed Spin surface. This TFT, called the *Arf theory*, was studied extensively in [**Gun16**].

3.4.0.2. *The Atiyah-Bott-Shapiro orientation revisited.* First let us recall from Section §3.2.3.1 that the Arf invariant of a closed Spin surface Σ may be constructed as a pushforward in ko-theory:

$$\pi_!^{\Sigma} 1_{\Sigma} \in ko^{-2}(\mathrm{pt}) \cong \mathbb{Z}/2.$$

As explained in §3.2.3.1, the pushforward map in ko-theory is constructed using the Atiyah-Bott-Shapiro orientation of ko, which may be encoded as a map of spectra

$$\widehat{A} : MTSpin \to ko.$$

In fact (as shown in [**Gun16**]) the entire Arf theory factors naturally through the Atiyah-Bott-Shapiro orientation:[7]

$$Z_A : |\overline{\mathsf{Bord}_2^{\mathrm{Spin}}}| \to MTSpin \xrightarrow{\widehat{A}} ko \xrightarrow{\mathit{Cliff}} |\mathsf{sAlg}_{\mathbb{C}}^{\times}|.$$

As explained in §3.2.3.2, the Arf-Brown invariant may also be interpreted as a pushforward in (twisted) KO-theory. One may reinterpret this as arising from the following twisted form of the Atiyah-Bott-Shapiro orientation:

$$MTPin^- \simeq MTSpin \wedge \Sigma^{-1} MO_1 \xrightarrow{\widehat{A} \wedge \mathrm{id}} ko \wedge \Sigma^{-1} MO_1. \tag{3.4.3}$$

The factor $\Sigma^{-1}MO_1$ is the Thom spectrum of the virtual vector bundle over BO_1 corresponding to the representation sphere $S^{\sigma-1}$.

[7]The map *Cliff* is so called because it takes an element of $ko^0(\mathrm{pt})$, represented by a finite dimensional vector space V, to the (complex) Clifford algebra associated to a positive definite inner product on V.

Note that the twisted Atiyah-Bott-Shapiro orientation induces an isomorphism

$$\pi_2(MTPin^-) \cong \pi_2(ko \wedge \Sigma^{-1} MO_1).$$

It follows that the Arf-Brown theory factors through the twisted ABS map (3.4.3):

$$|\overline{\mathsf{Bord}_2^{\mathrm{Pin}-}}| \to MTPin^- \to ko \wedge \Sigma^{-1} MO_1 \to |\mathsf{sAlg}_{\mathbb{C}}^\times|.$$

Remark 3.4.4. Freed-Hopkins [**FH16a**] define an involution on $\Sigma^2 MTPin_2^-$ obtained by taking a pin$^-$ structure to its opposite (obtained by tensoring with the orientation double cover), and an involution on $|\mathsf{sAlg}_{\mathbb{C}}^\times|$ induced by complex conjugation. The homotopy fixed point spectra are $(\Sigma^2 MTPin_2^-)^{h\mathbb{Z}/2} \simeq \Sigma^2 MTSpin_2$ and $|\mathsf{sAlg}_{\mathbb{C}}^\times|^{h\mathbb{Z}/2} \simeq |\mathsf{sAlg}_{\mathbb{R}}^\times|$. A 2D invertible pin$^-$ TFT is said to have *reflection structure* if the map of spectra $\Sigma^2 MTPin_2^- \to |\mathsf{sAlg}_{\mathbb{C}}^\times|$ it defines is equivariant with respect to these involutions [**FH16a**] (see also [**JF17**]).

The Arf-Brown theory naturally carries a reflection structure, which follows from the arguments of Freed-Hopkins [**FH16a**]. This is also hinted at by physics: in §3.5 we discuss how the $\mathbb{Z}/8$ classification of deformation classes of 2D invertible pin$^-$ TFTs is conjecturally linked to the $\mathbb{Z}/8$ classification of 2D pin$^-$ SPT phases. Some physics-based classifications of these SPTs [**FK11, GJF19**] are rooted in real superalgebra, tying the $\mathbb{Z}/8$ classification to the 8-fold periodicity of Morita equivalence classes of real Clifford algebras.

3.5. The time-reversal-invariant Majorana chain

The Arf-Brown TFT is believed to arise in physics as part of the classification of topological phases of matter. In this section, we discuss one of its conjectural appearances, as the low-energy theory of the Majorana chain with time-reversal symmetry, and some background on this occurrence.

3.5.1. Symmetry-protected topological phases. Condensed-matter theorists are interested in classifying topological phases of matter: given a dimension and a collection of symmetries to be preserved (called the *symmetry type*), what physical systems can occur, and what kind of data is needed to specify one up to a suitable notion of equivalence? This is a difficult problem in general, but can be simplified by restricting to nice subclasses of phases.

This problem is complicated by the lack of a mathematical definition of a topological phase. Nonetheless, arguments from physics suggest some properties that a definition will have: for example, given two topological phases with the same dimension and symmetry type, it should be possible to formulate them both on the same ambient space but with no interactions between them, creating another phase. This commutative monoid-like operation is called *stacking*.

Definition 3.5.1. A *symmetry-protected topological (SPT) phase* is a topological phase of matter which is invertible under stacking: after stacking with some other phase, it's equivalent to the trivial phase.[8]

Though this isn't a mathematical definition, it tells us that equivalence classes of SPTs should form an abelian group. The computation of this abelian group given a dimension and symmetry type has been the subject of considerable recent research activity at the interface of topology and physics (for a long list of references, see [**GJF19**, §1]).

Remark 3.5.2. The original definition in physics of a symmetry-protected phase is one which is inequivalent to the trivial theory when its symmetry type is considered, but which is equivalent in the absence of symmetry. According to this definition, the trivial phase is not an SPT, so the group structure is lost. Our interest in the group structure motivates us to allow the trivial phase.

To classify SPTs, one generally needs a model for phases of matter and equivalences between them.[9] Lattice models are a common choice: roughly speaking, an n-dimensional lattice model is a way of assigning to any closed n-manifold M with a simplicial structure the following data:

- a complex vector space $\mathcal{H}$ determined by local combinatorial data on M, called the *state space*; and
- a self-adjoint operator $H\colon \mathcal{H} \to \mathcal{H}$ also determined by local combinatorial data, called the *Hamiltonian*.

A lattice model is *gapped* if there is an $\varepsilon > 0$ such that as the simplicial structure is refined on any closed n-manifold M, the difference between the two smallest eigenvalues of H is greater than ε. Two gapped lattice models are equivalent if one can be deformed into the other through local deformations of $\mathcal{H}$ and H that preserve a gap in $\operatorname{Spec} H$.

The symmetry type corresponds to a choice of tangential structure on M, expressed in terms of the simplicial structure. Here are some examples.

- The default symmetry type fixes an orientation on M, expressed through a consistent local orientation of its simplices.
- A phase has *time-reversal symmetry* if we can choose M to be unoriented. In this case one doesn't need orientations on simplices. Alternatively, because lattice models are built from local data, one

[8]We haven't provided a definition of the trivial phase, and the definition will depend on one's model for topological phases. But it should have no interesting physics, and its partition functions on closed manifolds should all be equal to 1.
[9]We note, however, the existence of model-independent approaches [**GJF19, Xio18, XA18**].

can formulate the model on a simplicial disc, together with an explicit action of reflection on $\mathcal{H}$; this is how time-reversal symmetry is implemented for the Majorana chain.

- There is a notion of a *fermionic SPT* which is believed to correspond to spin structure; see §3.5.2 below.
- Given a finite group G, an *internal G-symmetry* corresponds to the data of a principal G-bundle $P \to M$. This can be formulated as a function from the 1-simplices of M to G encoding the monodromy of P or by placing a simplicial structure on P itself [**OMD16**].

These symmetries may interact in nontrivial ways: for example, there are two ways to implement time-reversal symmetry in fermionic phases, corresponding to pin$^+$ and pin$^-$ structures on M.

Remark 3.5.3. The above is not a rigorous mathematical definition of topological phases of matter. Providing a rigorous framework for this classification problem is a significant open problem in this field.

See [**Sab18**] for more about lattice models.

There are many approaches to classifying SPTs. We will use a low-energy limit approach, because it reduces modulo a conjecture to a completely mathematical problem, the classification of TFTs.

Definition 3.5.4. Given a gapped lattice model with Hamiltonian H, its space of *ground states* on a closed manifold M is the eigenspace for the smallest eigenvalue of H.

In examples, this depends on the underlying manifold but not its triangulation, behaving like a topological field theory. Conjecturally, it *is* (part of) a topological field theory:

Ansatz 3.5.5. Given a d-dimensional lattice model with symmetry type H_d, there is a fully extended, reflection positive $(d+1)$-dimensional TFT[10] Z with the same symmetry type, called the *low-energy (effective) field theory* of the lattice model, whose deformation class can be determined from the data of the lattice model, and such that

(1) if N is a closed d-manifold, $Z(N)$ is isomorphic to the space of ground states of the lattice model on N;

(2) if $\varphi\colon N \to N$ is a diffeomorphism and N_φ denotes its mapping torus, there is a well-defined action of φ on the ground states of the lattice model on N, and $Z(N_\varphi)$ is the trace of this action.

In addition,

[10]In general one must also allow TFTs tensored with an invertible, non-topological theory; see [**FH16a**, §5.4]. However, this will not come into play in this paper.

(3) deformation-equivalent lattice models should have deformation-equivalent low-energy effective field theories, and

(4) if S_0 and S_1 are lattice models with low-energy theories Z_0 and Z_1, respectively, the low-energy theory of $S_0 \otimes S_1$ should be $Z_0 \otimes Z_1$.

It is believed that the map sending a lattice model to its low-energy theory is surjective onto the set of deformation classes of fully extended, reflection positive $(d+1)$-dimensional H_d-TFTs.

For discussion of this prediction, see [**FH16a, Gai17, RW18**]; for discussion of reflection positivity in the invertible case, see [**FH16a**, §8.2]. For the rest of this section, we assume Ansatz 3.5.5.

Ansatz 3.5.5 implies in particular that the group of equivalence classes of d-dimensional SPTs with a given symmetry type is isomorphic to the group of deformation classes of reflection positive invertible $(d+1)$-dimensional TFTs with the same symmetry type, a fact which Freed-Hopkins [**FH16a**, §9.3] use to classify fermionic SPTs. This approach to classifying SPTs is also undertaken in [**Cam17, PWY17, SSR17a, DT18**].

3.5.1.1. *Context for the Majorana chain.* We now specialize to the group of 2d pin$^-$ SPTs, which is believed to be isomorphic to $\mathbb{Z}/8$.[11] This can be proven assuming Ansatz 3.5.5, as in [**FH16a**, (9.7.7)]: we saw in §3.3.3.4 that the group of deformation classes of 2d reflection positive invertible pin$^-$ TFTs is $[MTPin^-, \Sigma^3 I_{\mathbb{Z}}] \cong \mathbb{Z}/8$. Other approaches to this $\mathbb{Z}/8$ classification can be found in [**GW14, KTTW15, BWHV17, CSRL17, GJF19, SSR17a**].

The Majorana chain is a 2d fermionic SPT phase with time-reversal symmetry making it into a pin$^-$ phase, and several physical arguments have shown that it's the generator of the $\mathbb{Z}/8$ of such phases.[12] The Majorana chain was originally studied by Kitaev [**Kit01**] with an eye towards applications in quantum computing, then given time-reversal symmetry by Fidkowski-Kitaev [**FK10**] and Turner-Pollmann-Berg [**TPB11**], who observed that it generated a $\mathbb{Z}/8$ of SPTs. Therefore, Ansatz 3.5.5 implies that its low-energy field theory is a tensor product of an odd number of copies of the Arf-Brown theory. In what follows, we will formulate the Majorana chain on a pin$^-$ 1-manifold and study its low-energy behavior.

Remark 3.5.6. There's an additional way in which the Arf-Brown theory is expected to arise in physics. Though we won't discuss it in detail, we'll point the interested reader to some references.

Associated to a free fermion theory in dimension d is its anomaly theory, a $(d+1)$-dimensional invertible field theory of the same symmetry type. The group of equivalence classes of 1-dimensional free fermion

[11] From a mathematical perspective, because mathematical definitions for SPTs haven't been written down, this isn't yet a theorem. It's expected that once the definitions are in place, it will be true.

[12] Similarly, this is not yet a mathematical theorem because we don't have a mathematical definition of an SPT.

theories with pin$^-$ symmetry is conjecturally isomorphic to $\mathbb{Z}$ with the Majorana chain as a generator, and its anomaly theory is conjecturally the Arf-Brown TFT. For general free fermion systems, this conjecture is due to Freed-Hopkins [**FH16a**, §9.2.6]; Witten [**Wit16**, §5] provides a physical argument specifically for the time-reversal symmetric Majorana chain.

These two appearances of the Arf-Brown TFT from the Majorana chain are believed to be related: one can regard a free system as an interacting system with the same dimension and symmetry type, defining a group homomorphism from equivalence classes of free fermion theories to SPTs. For 2d pin$^-$ theories, this is believed to be the quotient map $\mathbb{Z} \to \mathbb{Z}/8$, a surprising fact first noticed by Fidkowski-Kitaev [**FK10, FK11**] and Turner-Pollmann-Berg [**TPB11**], and argued a different way by You-Wang-Oon-Xu [**YWOX14**]. Freed-Hopkins [**FH16a**, §§9.2, 9.3] provide a conjecture describing this homomorphism in general, then study it in several specific cases, including 2d pin$^-$ theories.

3.5.2. Combinatorial spin and pin structures. The Majorana chain is a fermionic SPT. This is believed to correspond to building the state space and Hamiltonian using superalgebra. In relativistic quantum field theory, the spin-statistics theorem implies such a system should be formulated on spin manifolds, but in the condensed-matter setting, the theorem doesn't apply. Nonetheless, it appears that spin structures are the correct setting for fermionic phase of matter, in that the data of a fermionic phase of matter depends on a choice of spin structure on the underlying manifold in examples [**GK16**].

Time-reversal symmetry can act on fermionic phases in two ways: by squaring to 1 or by squaring to the grading operator. The former is believed to correspond to a pin$^-$ structure on the underlying manifold, and the latter to a pin$^+$ structure [**KTTW15**].

The Majorana chain admits a time-reversal symmetry T squaring to 1, so to formulate the Majorana chain on a compact 1-manifold M with this symmetry, we must choose a pin$^-$ structure on M and encode it in the data of the lattice somehow. In general, this is somewhat tricky: for spin structures, this was solved by Cimasoni-Reshetikhin [**CR07**] in dimension 2 and Budney [**Bud13**] in all dimensions, but the analogue for pin$^-$ structures appears to be unknown. Since we're only studying 1-manifolds, we can use an explicit, simpler construction: there are two pin$^-$ structures on a closed interval relative to a fixed pin$^-$ structure on its boundary, so we will fix pin$^-$ structures on the vertices of M and use the data of the class of the pin$^-$ structure on the edge.

Fix, once and for all, a pin$^-$ point pt.

Definition 3.5.7. Let I be a closed interval and $\partial I = \{a, b\}$. Fix a pin$^-$ structure on ∂I and pin$^-$ isomorphisms $\mathrm{pt} \cong a$ and $\mathrm{pt} \cong b$. A *relative pin$^-$ structure* on I is a pin$^-$ structure on I which restricts to the specified pin$^-$ structure on ∂I. We consider two relative pin$^-$ structures on I equivalent if there's a pin$^-$ diffeomorphism between them covering the identity and respecting this data, i.e. it restricts to the identity on ∂I and intertwines the pin$^-$ diffeomorphisms with pt.

The pin$^-$ diffeomorphisms $a \cong \mathrm{pt} \cong b$ define a pin$^-$ structure on $I/\partial I$, and sending $I \mapsto I/\partial I$ defines an isomorphism from the set of equivalence classes of relative pin$^-$ structures on I to the set of diffeomorphism classes of pin$^-$ structures on S^1; let I_0 be a relative pin$^-$ structure on I which maps to S^1_{nb} and I_1 be one which maps to S^1_b.

Lemma 3.5.8. *Concatenation defines a group structure on the equivalence classes of relative pin$^-$ intervals. This group is isomorphic to $\mathbb{Z}/2$ and the generator is I_0.*

Proof. Because every 1-manifold M can be stably framed relative to a fixed stable framing on the boundary, we may define the pin$^-$ structures on I_0 and I_1 as those induced by framings. Specifically, I_0 is induced by the trivial framing (the restriction of the usual framing on $\mathbb{R}$ to $[0,1]$), and I_1 is induced by the nontrivial framing. Two concatenated copies of this framing are equivalent to the trivial framing when the endpoints are fixed (see [**DSPS13**, Remark 1.3.1]) and concatenating with the trivial framing does not change the equivalence class of framing on an interval, giving the claimed group structure. $\square$

Let M be a compact pin$^-$ 1-manifold with a simplicial structure, and let $\Delta^i(M)$ denote its set of i-simplices. For each $v \in \Delta^0(M)$, fix a pin$^-$ isomorphism $v \cong \mathrm{pt}$. Since the groupoid of pin$^-$ structures on a point is equivalent to $\bullet/(\mathbb{Z}/2)$, an isomorphism with pt is a choice. For each $e \in \Delta^1(M)$, the pin$^-$ structure on M defines a relative pin$^-$ structure on $\overline{e}$. Thus $e \cong I_j$ for some $j \in \{0,1\}$; define $t(e) \coloneqq j$. From the function $t\colon \Delta^1(M) \to \mathbb{Z}/2$ one can recover the pin$^-$ structure on M up to isomorphism. We will call t the *combinatorial pin$^-$ data* of M.

Lemma 3.5.9. *Let M be a pin$^-$ circle with a simplicial structure and m be the number of edges of e with $t(e) = 1$.*

- *If m is odd, $M \cong S^1_b$.*
- *If m is even, $M \cong S^1_{nb}$.*

Proof. Fix a vertex $v \in M$. Using the group law from Lemma 3.5.8, we can concatenate adjacent intervals for all vertices except v, resulting in a simplicial structure on M with a single vertex at v and a single edge e with $t(e) = m \bmod 2$. The result then follows from the definition of t. $\square$

3.5.3. Defining the Majorana chain. Let M be a compact pin$^-$ 1-manifold with a simplicial structure. Associated to each vertex $v \in \Delta^0(M)$, we associate a trivialized odd line $\mathbb{C}_v^{0|1}$ and define the local state space $\mathcal{H}_v \coloneqq \Lambda(\mathbb{C}_v)$. The state space for the Majorana chain on M is

$$\mathcal{H} \coloneqq \bigotimes_{v \in \Delta^0(M)} \mathcal{H}_v. \tag{3.5.10}$$

Let F denote the space of functions $\Delta^0(M) \to \mathbb{C}$, regarded as a purely odd vector space. Then $\mathcal{H} \cong \Lambda^*(F)$, and hence $\mathcal{H}$ is generated by the δ-functions δ_v for $v \in \Delta^0(M)$, where each δ_v is odd.

Definition 3.5.11. Let $v \in \Delta^0(M)$.

- The *annihilation operator* associated to v, denoted ι_v, is the interior product with δ_v.
- The *creation operator* associated to v, denoted ε_v, is the exterior product with δ_v.
- The *Majorana operators* associated to v are

$$\begin{aligned} c_v &\coloneqq \varepsilon_v + \iota_v \\ d_v &\coloneqq \varepsilon_v - \iota_v. \end{aligned}$$

Remark 3.5.12. The notation for the Majorana operators in [**Kit01, FK11**] corresponds to ours as follows: after ordering the vertices $v_1, \dots, v_n$ on an interval in the direction defined by the orientation, their c_{2j-1} is our c_{v_j}, and their c_{2j} is i times our d_{v_j}. In some papers, the Majorana chain is instead called the *Majorana wire* or *Kitaev chain.*

To define the Hamiltonian, we must orient M. This is a bit surprising, because the Majorana chain admits a time-reversal symmetry and therefore ought to make sense on a pin$^-$ manifold without using the fact that all 1-manifolds are orientable, but if we vary the orientation on M, we obtain a different Hamiltonian. We expect that the eigenspaces for the Hamiltonian, as subspaces of $\mathcal{H}$, end up not depending on the choice of orientation, but verifying this would require a different approach: ours uses a choice of orientation to construct a Clifford module isomorphic to $\mathcal{H}$, but without choosing an isomorphism, making it difficult to track the dependence on orientation.

The Hamiltonian for the Majorana chain is a sum of local terms for each edge. Fix an orientation on M, so that each edge e has an induced orientation; we write $\partial e = v - w$ to mean $\partial e = \{v, w\}$, and that, in the induced orientation on the boundary, v is the positively oriented boundary point and w is the negatively oriented one. For each $v \in \Delta^0(M)$, choose a pin$^-$ isomorphism $v \cong \mathrm{pt}$, and let $t\colon \Delta^1(M) \to \mathbb{Z}/2$ be the induced combinatorial pin$^-$ data. Then, the Hamiltonian on M is

$$H := \frac{1}{2} \sum_{\substack{e \in \Delta^1(M) \\ \partial e = v - w}} (-1)^{t(e)} c_v d_w. \tag{3.5.13}$$

Time-reversal symmetry acts on $\mathcal{H}$ as complex conjugation; since c_v and d_w are real, this commutes with the Hamiltonian, so the Majorana chain admits a time-reversal symmetry squaring to 1.

Remark 3.5.14. In physics, a Majorana fermion is a fermion which is its own antiparticle, meaning that its creation and annihilation operators coincide. Because the Clifford relations imply $c_v^2 = 1$ and $(i \cdot d_v)^2 = 1$, these operators can be interpreted as creating up to two Majorana fermions located at v. The Hamiltonian (3.5.13) is expressing a relationship between Majorana fermions at adjacent vertices: if $\partial e = v - w$, then the Hamiltonian specifies that low-energy states must have a relationship between the Majorana fermions corresponding to c_v and $i \cdot d_w$.

Because it would be interesting to observe a Majorana fermion, the Majorana chain has been studied experimentally [**MZF$^+$12, DYH$^+$12, DRM$^+$12, RLF12, FVHM$^+$13**]. To our knowledge, however, these experiments have not considered the behavior of the Majorana chain under stacking or time-reversal symmetry.

3.5.4. The low-energy TFT. We'd like to use Ansatz 3.5.5 to determine the deformation class of the low-energy theory Z of the Majorana chain, but it doesn't tell us everything. For example, neither pin$^-$ structure on $\mathbb{RP}^2$ is bordant to a disjoint union of mapping tori, so we won't be able to calculate $Z(\mathbb{RP}^2)$. Nonetheless, Ansatz 3.5.5 tells us we can compute the state space of any closed 1-manifold and the partition functions of all pin$^-$ tori and Klein bottles. In particular, we'll find that $Z(S^1_{nb})$ is an odd line, which is enough to imply that Z is one of the four generators of the $\mathbb{Z}/8$ of deformation classes of reflection positive 2d pin$^-$ invertible field theories.[13]

Let $\pi\colon M' \to M$ be the orientation double cover, and give M' the simplicial structure which makes π a simplicial map. The orientation of M induces an orientation of the 0-skeleton of M', M'_0, which is a compact oriented 0-manifold, so this orientation defines a function $\mathfrak{o}\colon M'_0 \to \{\pm 1\}$ sending a positively oriented point to 1 and a negatively oriented point to -1.

Let $n := |\Delta^0(M)|$.

[13] Since Ω_2^{Spin} and $\Omega_2^{\mathrm{Pin}^+}$ are generated by mapping tori, this ambiguity does not appear for 2d spin and pin$^+$ phases. For general symmetry types, however, this is not the case, and additional work is needed to uniquely determine the low-energy field theory of an SPT. This perspective is taken up by Shiozaki-Shapourian-Ryu [**SSR17b**].

Lemma 3.5.15. *The algebra generated by c_v and d_v is canonically isomorphic to $C\ell(\pi^{-1}(v), \mathfrak{o})$ and isomorphic to $C\ell_{1,1}$. The algebra generated by all Majorana operators is canonically isomorphic to $C\ell(M_0', \mathfrak{o})$, and noncanonically isomorphic to $C\ell_{n,n}$.*

PROOF. If $V \in \Delta^0(M)$, let v_+, (resp. v_-) be the positively (resp. negatively) oriented preimage of v. We define the maps $\langle c_v, d_v\rangle \to C\ell(\pi^{-1}(v), \mathfrak{o})$ and $\langle c_w, d_w \mid w \in \Delta^0(V)\rangle \to C\ell(M_0', \mathfrak{o})$ to send $c_v \mapsto v_+$ and $d_v \mapsto v_-$. For this to define an isomorphism of algebras, one must check the defining relations of the Clifford algebra: $c_v^2 = 1$, $d_v^2 = -1$, $[c_v, d_v] = -1$, and if $v \neq w$, $[c_v, c_w] = [d_v, d_w] = [c_v, d_w] = -1$. These follow directly from the definition of the Majorana operators.

Since $\mathfrak{o}|_{\pi^{-1}(v)}$ sends $v_+ \mapsto 1$ and $v_- \mapsto -1$, $C\ell(\pi^{-1}(v), \mathfrak{o}) \cong C\ell_{1,1}$ and $C\ell(M_0', \mathfrak{o}) \cong C\ell_{n,n}$, the latter after choosing an ordering of the vertices of v. $\square$

Let M be a spin circle with a simplicial structure, and let $t\colon \Delta^1(M) \to \mathbb{Z}/2$ be the combinatorial data associated to it. Let $n := |\Delta^0(M)|$; then, the state space $\mathcal{H}$ is a $\mathbb{Z}/2$-graded $C\ell(M_0', \mathfrak{o})$-module.

Theorem 3.5.16 ([ABS64, §5])**.** *Up to isomorphism, $C\ell(M_0', \mathfrak{o})$ has a single irreducible module M, which is 2^n-dimensional. Up to even isomorphism, $C\ell(M_0', \mathfrak{o})$ has two irreducible supermodules, both isomorphic to M after forgetting the $\mathbb{Z}/2$-grading, and they are parity changes of each other.*

Since $\dim \mathcal{H} = 2^n$, then $\mathcal{H}$ is one of the two irreducible $C\ell(M_0', \mathfrak{o})$-supermodules. The Hamiltonian acts on $\mathcal{H}$ as an element of $C\ell(M_0', \mathfrak{o})$, since it's a sum of products of Clifford generators. Thus, to compute its spectrum, it suffices to compute the action of $H \in C\ell(M_0', \mathfrak{o})$ on any irreducible $C\ell(M_0', \mathfrak{o})$-module A. To determine the parity of the space of ground states, we need to know whether $\mathcal{H}$ is graded isomorphic to A or ΠA, which we will do by fixing a grading operator $\varepsilon \in C\ell(M_0', \mathfrak{o})$ and comparing its action on $\mathcal{H}$ and on A.

Lemma 3.5.17. *There is a unique isomorphism of superalgebras $\varphi\colon C\ell_{1,1} \xrightarrow{\cong} \mathrm{End}(\mathbb{C}^{1|1})$ sending*

$$v_+ \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad v_- \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.5.18}$$

PROOF. One can directly verify that $\varphi(v_\pm)$ are odd, $\varphi(v_\pm)^2 = \pm I$, and $\varphi(v_+)$ anticommutes with $\varphi(v_-)$. Uniqueness follows because v_+ and v_- generate $C\ell_{1,1}$. $\square$

Thus, $g := \varphi(v_+ v_-)$ is a grading operator on $\mathbb{C}^{1|1}$.

Let $e \in \Delta^1(M)$ with $\partial e = v - w$. Since ∂e is an oriented 0-manifold, it comes with a function $\mathfrak{o}_e\colon \partial e \to \{\pm 1\}$; the algebra generated by c_v and d_w is canonically isomorphic to $C\ell(\partial e, \mathfrak{o}_e)$, which is isomorphic to $C\ell_{1,1}$. Let $C\ell(\partial e, \mathfrak{o}_e)$ act on a $\mathbb{C}^{1|1}$ through the isomorphism from Lemma 3.5.17, and call it $\mathbb{C}_e^{1|1}$. Then there's a canonical isomorphism

$$C\ell(M_0', \mathfrak{o}) \cong \bigotimes_{e\in\Delta^1(M)} C\ell(\partial e, \mathfrak{o}_e), \tag{3.5.19}$$

so $C\ell(M_0', \mathfrak{o})$ acts on

$$A \coloneqq \bigotimes_{e\in\Delta^1(M)} \mathbb{C}_e^{1|1}, \tag{3.5.20}$$

making A into a graded $C\ell(M_0', \mathfrak{o})$-module. Since $\dim_{\mathbb{C}} A = 2^n$, A must be irreducible.

Let m be the number of edges e of M with $t(e) = 1$.

Proposition 3.5.21. *Let $V \subset A$ denote the eigenspace for the smallest eigenvalue of H acting on A. Then V is one-dimensional, and has parity $n - m \bmod 2$.*

PROOF. Let $a \in A$ be a pure tensor of homogeneous elements of $\mathbb{C}_e^{1|1}$ as e ranges over the edges of M, and let $|a|$ denote its degree. For any $e \in \Delta^1(M)$, we let $|a|_e$ be 1 if the component of a from $\mathbb{C}_e^{1|1}$ is odd, and 0 otherwise.

If $\partial e = v - w$, then $c_v d_w$ acts by the grading operator g_e on $\mathbb{C}_e^{1|1}$, and therefore the action of H is

$$H = \frac{1}{2} \sum_{e\in\Delta^1(M)} (-1)^{t(e)} \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes g_e \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}. \tag{3.5.22}$$

It suffices to describe the action of H on pure tensors of homogeneous elements, so let a be such a tensor. If $t(e) = 0$ for all edges e, then H differs from $(n/2) \cdot \mathrm{id}$ on a by subtracting 1 for each odd component of a. Therefore H acts on a as $(n - 2|a|)/2$, in which case the ground states are the top-degree vectors, with eigenvalue $-n/2$.

More generally, if e is an edge with $t(e) = 1$, it contributes $-g_e$ to H instead of g_e. This change is equivalent to multiplying by $2(2|a|_e - 1)$, so if $e_1, \dots, e_m$ are the edges with $t(e_i) = 1$, then the action of the Hamiltonian on a is

$$H \cdot a = (n - 2k + 2(2|a|_{e_1} - 1) + \cdots + 2(2|a|_{e_m} - 1))\frac{a}{2}. \tag{3.5.23}$$

The a which minimize the eigenvalue are those whose component in $\mathbb{C}_e^{1|1}$ is odd if $t(e) = 0$ and even if $t(e) = 1$; these form a one-dimensional vector space with parity $n - m$. $\square$

Proposition 3.5.24. *Let*

$$\varepsilon \coloneqq \prod_{v\in\Delta^0(M)} d_v c_v \in C\ell(M_0', \mathfrak{o}).$$

(The Clifford relations imply this doesn't depend on the order of the vertices in the product.) Then

- *on $\mathcal{H}$, ε acts on a homogeneous degree-k element by multiplication by $(-1)^{n-k}$, and*
- *on A, ε acts on a homogeneous degree-k element by multiplication by $(-1)^{k-1}$.*

Proof. On $\mathcal{H}$, ε acts as

$$(\varepsilon\cdot) = \prod_{v\in\Delta^0(M)} (\varepsilon_v - \iota_v)(\varepsilon_v + \iota_v) = \prod_v (\varepsilon_v \iota_v - \iota_v \varepsilon_v). \tag{3.5.25}$$

It suffices to understand how this acts on pure wedges $\omega = \lambda\delta_{v_{i_1}} \wedge \cdots \wedge \delta_{v_{i_\ell}}$. On ω, $\varepsilon_v\iota_v - \iota_v\varepsilon_v$ acts by the identity if $v = v_{i_j}$ for some j, and by -1 otherwise. Therefore $\varepsilon \cdot \omega = (-1)^{n-k}\omega$.

To compute the action of ε on A, we rearrange it into a more convenient form. Choose a $v_1 \in \Delta^0(M)$, and let $v_2, \dots, v_n$ be the vertices encountered in order as one traverses the positively oriented path around M starting at v_1. Thus for each i, there's an edge e_i with $\partial e_i = v_{i+1 \bmod n} - v_i$. Then,

$$\varepsilon = d_{v_1} c_{v_1} \cdots d_{v_n} c_{v_n} = (-1)^n c_{v_1} d_{v_1} \cdots c_{v_n} d_{v_n}. \tag{3.5.26}$$

Since this string has n letters, reversing it is a permutation of parity $(-1)^n$:

$$= d_{v_n} c_{v_n} \cdots d_{v_1} c_{v_1}. \tag{3.5.27}$$

Finally, we commute c_{v_1} past the remaining $2n-1$ operators:

$$= -\underbrace{c_{v_1} d_{v_n}}_{g_n} c_{v_n} \cdots \underbrace{c_3 d_2}_{g_2}\underbrace{c_2 d_1}_{g_1}. \tag{3.5.28}$$

Therefore ε acts by -1 times the usual grading operator on $\mathbb{C}^{n|n}$ (i.e. the one which is -1 on odd states). ☐

Corollary 3.5.29. *As graded $C\ell(M_0', \mathfrak{o})$-modules, $\mathcal{H} \cong \Pi^{n-1} A$, so the ground states of the Majorana chain on M are*

- *an even line if m is odd (so $M \cong S^1_b$), and*
- *an odd line if m is even (so $M \cong S^1_{nb}$).*

Proof. In Proposition 3.5.21, we saw that the ground states of H acting on A have parity $n - m \bmod 2$, but by Proposition 3.5.24 the difference in the parities of $\mathcal{H}$ and A is $n - 1 \bmod 2$. Hence the ground state space of H acting on $\mathcal{H}$ has parity $n - m - (n-1) = m - 1$. ☐

The parity of the ground states on S^1_b and S^1_{nb} is calculated in a different way by Shapourian-Shiozaki-Ryu [**SSR17a**, Appendix D].

Corollary 3.5.30. *Assuming Ansatz 3.5.5, the low-energy TFT Z of the Majorana chain is a generator of the $\mathbb{Z}/8$ of deformation classes of reflection positive pin$^-$ invertible field theories. In particular, its deformation class is an odd multiple of the class of the Arf-Brown theory.*

Proof. By a result of Schommer-Pries [**SP18**, Theorem 11.1], we know Z is invertible, since there is a pin$^-$ structure on S^2 and $Z(S^1_b)$ and $Z(S^1_{nb})$ are both invertible in $\mathsf{sVect}_{\mathbb{C}}$. Since Z_{AB} generates the $\mathbb{Z}/8$ of deformation classes of reflection positive 2d pin$^-$ invertible TFTs, Z is deformation equivalent to $(Z_{AB})^{\otimes k}$ for some k, and is a generator iff k is odd.

Because $Z_{AB}(S^1_{nb})$ is an odd line, then $(Z_{AB})^{\otimes k}(S^1_{nb})$ has the same parity as k. Since $Z(S^1_{nb})$ is odd, then k is odd. □

We can also study the Majorana chain on pin$^-$ 1-manifolds with boundary, though again the Hamiltonian depends on an orientation. Kitaev [**Kit01**] found that the space of ground states on an interval I is two-dimensional; from the low-energy perspective, this follows from the fact that for any choice of pin$^-$ structure on I, $Z(I)$ is isomorphic to $C\ell_1$ as a $(C\ell_1, C\ell_1)$-bimodule. We can also see this directly from the lattice.

Suppose $n \coloneqq |\Delta^0(I)|$. Orient I and let $\partial I = v - w$. Then, c_w and d_v do not appear in the Hamiltonian on I. Since each term in H is $\pm 1/2$ times two Clifford generators not equal to c_w or d_v, both c_w and d_v commute with H, and therefore the algebra they generate, isomorphic to $C\ell_{1,1}$, acts on all eigenspaces of H. In particular, if V denotes the ground states of H, V is a $C\ell_{1,1}$-module, and by Theorem 3.5.16 is determined up to isomorphism by its dimension, which is even.

We can identify it with $C\ell_1$ in a manner similar to the proof of Proposition 3.5.21: define A in the same manner as above, except that we pretend there's an extra edge e_∂ joining v and w, so A is a $C\ell(I_0', \mathfrak{o})$-module, where I_0' is the 0-skeleton of the orientation double cover $I' \to I$ and $\mathfrak{o}\colon I_0' \to \{\pm 1\}$ is induced from the orientation as before. If H_{S^1} denotes the Hamiltonian from (3.5.22) (for the circle), then our Hamiltonian is

$$H = H_{S^1} - \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes g_{e_\partial}, \tag{3.5.31}$$

(where $t(e_\partial) \coloneqq 0$), whose action on a pure tensor of homogeneous elements $a \in A$ is

$$H \cdot a = (n - 2k + 2(2|a|_{e_1} - 1) + \cdots + 2(2|a|_{e_m} - 1) + |a|_{e_\partial})\frac{a}{2}. \tag{3.5.32}$$

Thus the ground state is two-dimensional, spanned by a pure tensor whose components are odd for all edges with $t(e) = 0$ and even otherwise, and a pure tensor whose components are odd for all edges with $t(e) = 0$ except e_∂, and even otherwise. Since $C\ell_1$ is the unique two-dimensional irreducible (ungraded) $C\ell_{1,1}$-representation up to isomorphism, the space of ground states on I is isomorphic to either $C\ell_1$ or $\Pi C\ell_1$. An argument similar to Proposition 3.5.24 shows that we get the former. Finally, to match the left $C\ell_{1,1}$-module description of the space of ground states with the $(C\ell_1, C\ell_1)$-bimodule description of $Z(I)$, recall that a left $C\ell_{-1}$-action on a module M is equivalent data to a right $C\ell_1$-action on M, which implies the space of ground states on I is $C\ell_1$ as a $(C\ell_1, C\ell_1)$-bimodule, in accordance with the calculation using the low-energy TFT.

CHAPTER 4

Invertible phases for mixed spatial symmetries and the fermionic crystalline equivalence principle

The content of this chapter appears on the ArXiv as the preprint [**Deb21a**]. It has been slightly edited to be streamlined with the rest of the thesis.

4.0. Introduction

The classification of topological phases of matter has been the subject of intensive research in condensed-matter physics and nearby areas of mathematics for the last decade, but difficult problems still remain: for example, there is not yet an accepted mathematical definition of a topological phase of matter, so researchers must study these systems using ansatzes or heuristic definitions of phases. Restricting to invertible phases, also known as *symmetry-protected topological (SPT) phases*, simplifies the classification question, but defining these phases precisely is also still an open problem. Freed-Hopkins [**FH16a**] make an ansatz modeling SPT phases using reflection positive invertible field theories (IFTs), then classify these IFTs using homotopy theory. This approach has been successfully employed in several cases to study examples of SPTs, as in [**FH16a, Cam17, WWW18, FHHT20, GOP$^+$20, PW21**].

Condensed-matter physicists are also interested in invertible phases in more general settings, including invertible phases on a particular space Y, as in [**Ran10**], or invertible phases symmetric for a group G acting on space, such as phases on the plane which have a rotation symmetry and the examples in [**SMJZ13**]. These spatial symmetries are often present in real-world examples of topological phases of matter (see [**WACB16, MYL$^+$17**] for one example), and can be modeled by lattice Hamiltonian systems in which the symmetry group also acts on the lattice, though again providing precise definitions is still open. In the case where G is a crystallographic group acting on $Y = \mathbb{R}^d$, these systems are called *crystalline SPT phases*. Freed-Hopkins' field-theoretic approach does not directly generalize to this setting, but there is a general ansatz of Kitaev [**Kit13a, Kit15**] that groups of phases on Y for a fixed symmetry type should define a generalized homology theory. Freed-Hopkins [**FH19a**] apply this to propose a classification of invertible phases in the presence of a G-action on space using equivariant generalized homology.

Researchers interested in computing groups of crystalline SPT phases provide *crystalline equivalence principles*, including the first such proposal of Thorngren-Else [**TE18**] and subsequent work in [**JR17, CW18, FH19a, ZWY⁺20, ZYQG20**]. Crystalline equivalence principles are arguments that groups of crystalline SPT phases are isomorphic to groups of ordinary SPT phases, where the symmetry type is modified. The theory is well-understood for symmetry types such as O_n and SO_n, corresponding to the physicists' notion of bosonic SPT, but for fermionic SPTs, corresponding to symmetry types such as Spin_n, Spin_n^c, $\mathrm{Pin}_n^{\pm}$, etc., the story is more complicated. Cheng-Wang [**CW18**], Zhang-Wang-Yang-Qi-Gu [**ZWY⁺20**], and Zhang-Yang-Qi-Gu [**ZYQG20**] study examples of fermionic crystalline SPTs, and show cases of a fermionic crystalline equivalence principle. Crucially, their work implies any fermionic crystalline equivalence principle must address fermionic phases in which the spatial symmetry mixes with fermion parity, which goes beyond the scope of Freed-Hopkins' ansatz.

The purpose of this paper is to formulate and prove such a fermionic crystalline equivalence principle (FCEP). To do so, we provide an ansatz expressing groups of invertible phases on a G-space Y in which the symmetry type can be merely locally constant over space and can mix with G, including as a special case spatial symmetries mixing with fermion parity. Given data $\mathcal{L}$ expressing this mixing and variance of the symmetry type, we define *phase homology* groups of Y, denoted $Ph_*^G(Y,\mathcal{L})$, and our ansatz predicts that the group of such invertible phases is isomorphic to $Ph_0^G(Y,\mathcal{L})$. Providing this ansatz is an additional goal of this paper, and is necessary input for our FCEP: the ansatz reexpresses the FCEP as an isomorphism between certain phase homology groups and groups of IFTs, as we state and prove in Theorem 4.2.8. This is the first homotopy-theoretic account of an FCEP, and to the best of our knowledge is the first fully general version of the FCEP in the literature.

As a corollary of the FCEP, the computation of phase homology groups that represent groups of point-group-equivariant fermionic phases reduces to computations of bordism groups; this paper's third goal is to make these computations in several examples, both for the purpose of testing our ansatz by comparing it to established predictions in physics, and for making additional predictions of groups of crystalline SPT phases in as yet unstudied settings. For symmetry types that have been studied before by other methods, our computations agree with the literature, bolstering our ansatz.

Now we go into a little more detail about these ansatzes and theorems. Freed-Hopkins [**FH19a**] formulate an ansatz for invertible phases of matter on a topological space Y equipped with an action of a compact Lie group G. First, specify the *symmetry type* of the theory as a map $\rho\colon H \to \mathrm{O}$, where $\mathrm{O} \coloneqq \varinjlim_n \mathrm{O}_n$ is the infinite orthogonal group and H is a topological group. From this data we can form a Madsen-Tillmann spectrum MTH, whose homotopy groups compute the bordism groups of manifolds with an H-structure on the tangent bundle. Let $I_{\mathbb{Z}}$ denote the Anderson dual of the sphere spectrum and $E \coloneqq \mathrm{Map}(MTH, \Sigma^2 I_{\mathbb{Z}})$.

Ansatz 4.0.1 (Freed-Hopkins [**FH19a**, Ansatz 3.3])**.** The abelian group of isomorphism classes of phases on Y equivariant for a G-symmetry that does not mix with the symmetry type H is the Borel-equivariant Borel-Moore homology group $E^{hG}_{0,\mathrm{BM}}(Y)$.

We will define equivariant Borel-Moore homology in the generality we need in Definition 4.1.14.

When G is trivial and $Y = \mathbb{R}^n$, the group of phases in Ansatz 4.0.1 is naturally isomorphic to $[MTH, \Sigma^{d+2} I_{\mathbb{Z}}]$, which Freed-Hopkins [**FH16a**] show is the classification of invertible field theories with symmetry type H.[1] When $Y = \mathbb{R}^d$ and G is a crystallographic group, this group of phases is expected to model the classification of crystalline SPT phases with this symmetry type, and indeed, Freed-Hopkins [**FH19a**, Example 3.5] prove a version of the bosonic crystalline equivalence principle of Thorngren-Else [**TE18**] as a consequence of their ansatz, matching physicists' predictions.

For fermionic phases, Ansatz 4.0.1 is not the full answer, and providing the full answer is a major goal of this paper. Physicists distinguish between phases with "spinless fermions" and "spin-1/2 fermions", asking how the spatial symmetry group G mixes with fermion parity. For example, one could consider phases on the plane equivariant for a C_4 rotation symmetry, and either ask that fermions' spin is unaffected by the spatial rotations, or that a full spatial rotation flips the spin on the fermion. This is reminiscent of the better-understood dichotomy of fermionic phases with a time-reversal symmetry T: one may have $T^2 = 1$ or T^2 equal to the fermion parity operator. These two classes of phases are modeled with different symmetry types, and similarly we use different data to model crystalline phases with spinless vs. spin-1/2 fermions.

To accommodate this mixing between the internal symmetry type H and the spatial symmetry group G, we generalize Freed-Hopkins' setup slightly using parametrized homotopy theory, considering local systems f of symmetry types over the base Y. These give rise to local systems of Thom spectra; if Y has a G-action we obtain from f a local system $\mathcal{L}'$ of Borel-equivariant Thom spectra, modeled as a functor from Y, though of as an ∞-groupoid, to the ∞-category Sp^G of Borel-equivariant spectra. Let $\mathcal{L} \coloneqq \mathrm{Map}(-, \Sigma^2 I_{\mathbb{Z}}) \circ \mathcal{L}'$ as maps $Y \to \mathsf{Sp}^G$, where $I_{\mathbb{Z}}$ has trivial G-action. We define the *equivariant phase homology* $Ph^G_*(Y; f)$ to be the equivariant Borel-Moore homology of the local system $\mathcal{L} \colon Y \to \mathsf{Sp}^G$.

Ansatz 4.1.19. The group of G-equivariant invertible phases on Y for this data is isomorphic to the equivariant phase homology group $Ph^G_0(Y; f)$.

[1]This result is conditioned on a conjecture about non-topological invertible theories; at present, we have as a theorem only that the invertible TFTs are classified by the torsion subgroup of this group. This is discussed by Freed-Hopkins [**FH16a**, §5.4] and Freed [**Fre19**, Lecture 9].

When f is trivializable, this reduces to Ansatz 4.0.1; in general, though, it allows the symmetry type to mix with the spatial symmetry, or to be merely locally constant on Y.

Now we specialize to the cases of spinless and spin-1/2 fermions. For spinless fermions, G and H do not mix, so we use the data of a constant local system of symmetry types and recover Freed-Hopkins' original ansatz. For spin-1/2 fermions, we specify data of an extension of G by H

$$0 \longrightarrow H \longrightarrow \widetilde{H} \longrightarrow G \longrightarrow 0, \tag{4.0.2}$$

together with a representation $\lambda\colon G \to \mathrm{O}_d$ dictating how G acts on space.[2] In the cases we consider in this paper, $H = \mathrm{Spin}$ or $H = \mathrm{Spin}^c$, and we specify $\widetilde{H}$ by way of the central extension

$$0 \longrightarrow \mu_2 \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 0 \tag{4.0.3}$$

whose isomorphism class is picked out by $w_2(V_\lambda) + w_1(V_\lambda)^2 \in H^2(BG;\mu_2)$, where $V_\lambda \to BG$ is the associated vector bundle to the representation λ and μ_2 is the group of square roots of unity. Then, $\widetilde{H} \coloneqq H \times_{\mu_2} \widetilde{G}$. Using this data, we build an equivariant local system f of symmetry types, obtaining a phase homology group $Ph_0^G(\mathbb{R}^d, f)$ that we predict is isomorphic to the group of invertible phases for this data.

The FCEP, previously studied in special cases by [**CW18, TE18, ZWY$^+$20, ZYQG20**], identifies groups of crystalline SPT phases with groups of fermionic SPT phases with an internal G-symmetry — but exchanging symmetry types: spinless crystalline phases correspond to spin-1/2 internal phases, and vice versa. Freed-Hopkins [**FH16a**] model groups of SPT phases with an internal G-symmetry using IFTs, and following Freed-Hopkins [**FH16a**] and the excellent overview by Beaudry-Campbell [**BC18**], these groups of TFTs can be expressed in terms of bordism groups of certain Thom spectra. Standard techniques in algebraic topology, notably the Adams spectral sequence over $\mathcal{A}(1)$, can be used to compute these bordism groups, so one application of a general version of the FCEP is to provide access to tractable tools for computing groups of crystalline SPT phases.

One of the major aims of this paper is to state and prove as a theorem a version of the FCEP, identifying phase homology groups with groups of IFTs; then Ansatz 4.1.19 translates this into a statement about crystalline SPTs and ordinary SPTs. In Definitions 4.2.3 and 4.2.4, we define the symmetry types for spinless and spin-1/2 fermions for a purely internal G-symmetry. In general these definitions are a little technical, but when the spatial representation λ factors through $\mathrm{SO}_d \subset \mathrm{O}_d$, the spinless internal symmetry type is

[2] We also specify some additional data; see Data 4.2.1 in §4.2 for the full details.

$H \times G \to \mathrm{O}$ and the spin-1/2 symmetry type is $H \times_{\mu_2} \widetilde{G} \to \mathrm{O}$, with the maps induced by the projection onto the first factor.

Theorem 4.2.8 (Fermionic crystalline equivalence principle). *Fixing data of G, H, λ, etc. as above, let $f_0, f_{1/2}$ denote the local systems of symmetry types for the case of spinless, resp. spin-1/2 fermions. Then $Ph_0^G(\mathbb{R}^d; f_0)$ is isomorphic to the group of deformation classes of d-dimensional IFTs for the spin-1/2 internal symmetry type, and $Ph_0^G(\mathbb{R}^d, f_{1/2})$ is isomorphic to the group of deformation classes of d-dimensional IFTs for the spinless internal symmetry type.*

The proof has two key steps.

(1) Phase homology groups are defined using equivariant parametrized homotopy theory. Proposition 4.1.29 reexpresses them using ordinary homotopy theory, as homotopy groups of a Thom spectrum built from a virtual vector bundle over $B\widetilde{H}$. The proof uses the Ando-Blumberg-Gepner-Hopkins-Rezk [**ABG+14a, ABG+14b**] approach to Thom spectra.

(2) Then, in Theorems 4.2.11 and 4.2.24, we "shear" this Thom spectrum, writing down a map $\widetilde{H}_n \to H_{n+d} \times G$ and showing that it induces a homotopy equivalence on Thom spectra, implying that phase homology groups are determined by H-bordism groups of a Thom spectrum over BG. Our proof is modeled on a fairly general shearing theorem in Freed-Hopkins [**FH16a**, §10].

After these two steps, the proof of Theorem 4.2.8 amounts to looking at the Thom spectra for the internal symmetry types and noticing that we end up with equivalent Thom spectra over BG in the cases we want to equate.

With this tool in hand, we can compute phase homology groups for point groups acting on $\mathbb{R}^d$, which are our model for groups of fermionic phases equivariant for point group symmetries. We do these computations for many 2d and 3d point groups, for both spinless and spin-1/2 fermions, and in Altland-Zirnbauer classes D and A (corresponding to $H = \mathrm{Spin}$, resp. Spin^c). Our computations use two avatars of the Adams spectral sequence. It is well-known that low-dimensional spin bordism can be computed using connective ko-homology and the Adams spectral sequence over $\mathcal{A}(1)$, and there is an excellent introduction to this technique by Beaudry-Campbell [**BC18**], but we also use a variant, computing spin^c bordism via ku-homology and the Adams spectral sequence over $\mathcal{E}(1)$, e.g. in §4.4.4.3. This is hardly a new idea, but there appear to be no examples of this specific kind of computation in the literature before now. We hope that our computations serve as useful examples of how to use this version of the Adams spectral sequence for spin^c bordism; this could be of independent interest.

For 2d point groups, these phases have been studied in the physics literature using very different methods. We compare our results with those of other researchers in §4.4.1.4, §4.4.2.4, §4.4.3.4, and §4.4.4.5, and find agreement, providing evidence in favor of Freed-Hopkins' ansatz and our generalization. However, there is not yet work on fermionic crystalline SPT phases for most 3d point groups, so our computations are predictions. We do many computations and make many predictions, and in §4.3.1 we collect a few that we think are relatively interesting or accessible. For example:

THEOREM. *Let A_4 act on $\mathbb{R}^3$ as the orientation-preserving symmetries of a tetrahedron. Then $Ph_0^{A_4}(\mathbb{R}^3, f)$ vanishes, where f is the local system of symmetry types for either spinless or spin-$1/2$ fermions in both Altland-Zirnbauer classes D and A.*

This is a combination of Theorems 4.5.4, 4.5.6 and 4.5.8. Therefore, assuming Ansatz 4.1.19, there are no nontrivial spinless nor spin-1/2 fermionic SPT phases equivariant for a chiral tetrahedral symmetry in Altland-Zirnbauer classes D or A. It would be interesting to see this prediction studied using lattice methods for fermionic crystalline phases.

In §4.6, we leave behind the FCEP and consider a different class of examples, SPTs equivariant for a glide reflection symmetry, providing a test for Freed-Hopkins' ansatz for a crystallographic group that is not a point group. Lu-Shi-Lu [**LSL17**] conjecture a general classification of these SPTs: that if $\mathrm{TP}_d(H)$ denotes the group of d-dimensional SPT phases with symmetry type H, then the group of d-dimensional glide SPTs is isomorphic to $\mathrm{TP}_{d-1}(H)\otimes\mathbb{Z}/2$. Xiong-Alexandradinata [**XA18**] derive this classification using physics-based arguments. We use Freed-Hopkins' ansatzes [**FH16a, FH19a**] to translate Lu-Shi-Lu's conjecture into a statement about phase homology groups and prove it.

Recall $E \coloneqq \mathrm{Map}(MTH, \Sigma^2 I_{\mathbb{Z}})$ and let $\widehat{Ph}_*^{\mathbb{Z}}(\mathbb{R}^d, \underline{E})$ denote the kernel of the forgetful map from $\mathbb{Z}$-equivariant phase homology to nonequivariant phase homology, where $\mathbb{Z}$ acts on $\mathbb{R}^d$ by glide translations, and $\underline{E} \to \mathbb{R}^d$ is the constant local system. This kernel models Lu-Shi-Lu's group of glide SPTs, as they require glide SPTs to be trivial in the absence of the glide symmetry.

THEOREM 4.6.4. *There is a natural isomorphism $\widehat{Ph}_0^{\mathbb{Z}}(\mathbb{R}^d; \underline{E}) \cong E_{-(d-1)} \otimes \mathbb{Z}/2$.*

This provides additional evidence in favor of the ansatz.

We want to mention that there are other homotopy-theoretic approaches to the study of phases of matter with a spatial symmetry, including those of Antolín Camarena, Sheinbaum, and collaborators [**AACSS16, SC20**] and Cornfeld-Carmeli [**CC21**]. These authors deal with free fermion phases, which are out of scope of this paper, though see §4.7.1.

4.0.1. Reader's guide to the different sections. Overview:

- In §§4.1–4.2 we discuss general aspects of our model for phases on a G-space Y and prove the FCEP. These sections involve the most homotopy theory.
- In §§4.3–4.5 we make phase homology calculations which according to Ansatz 4.1.19 calculate groups of fermionic crystalline SPT phases for which the symmetry group is a point group. We collect the results of these computations in Tables 1, 2, 3, 4, 5 and 6, and summarize the methods of computation in §4.3.2.
- In §4.6 we consider phases on $\mathbb{R}^d$ with a glide symmetry, and prove a theorem computing the corresponding phase homology classification.

Now a little more detail. In §4.1, we use Borel-equivariant parametrized homotopy theory to state a mild generalization of Freed-Hopkins' ansatz on invertible phases with spatial symmetry. In §4.1.1, we consider phases on a space Y without a group action, using local systems of symmetry types (Definition 4.1.4). We define phase homology and in Ansatz 4.1.7 express the group of invertible phases for such a local system in terms of phase homology. This is a slight generalization of [**FH19a**, Ansatz 2.1]. In §4.1.2, we allow group actions, defining equivariant local systems of symmetry types and equivariant Borel-Moore homology for a local system for the purpose of formulating Ansatz 4.1.19 expressing groups of invertible phases for a spatial symmetry in terms of equivariant phase homology. This is a minor generalization of Freed-Hopkins' ansatz [**FH19a**, Ansatz 3.3] to the parametrized setting. Then, in §4.1.3, we specialize to the case relevant to the FCEP, defining the local systems of symmetry types for spatial symmetries that mix with fermion parity. We prove Proposition 4.1.29 expressing the phase homology groups for this data in terms of nonequivariant, nonparametrized homotopy theory, and do not need equivariant or parametrized homotopy theory in the rest of the paper.

Next, §4.2, whose goal is to state and prove the FCEP. We begin in Definitions 4.2.2, 4.2.3 and 4.2.4 by defining the spinless and spin-1/2 local systems of symmetry types for both equivariant (i.e. G acting on space) and internal (G not acting on space) symmetries, and use these definitions to state our FCEP theorem in Theorem 4.2.8, identifying phase homology groups for these local systems in terms of groups of IFTs. As mentioned, the nontrivial part of the proof runs a shearing argument to simplify a Thom spectrum over $B\widetilde{H}$ into a smash product of MTSpin and a Thom spectrum over BG. In §4.2.1, we prove Theorem 4.2.11 accomplishing this in class D, for which $H = \mathrm{Spin}$. Then, in §4.2.2, we prove Theorem 4.2.24, which is the analogous theorem in class A, i.e. for $H = \mathrm{Spin}^c$, via a similar proof. Finally, in §4.2.3, we combine these arguments to prove Theorem 4.2.8.

In §4.3, we address a few generalities related to the FCEP before studying it in examples. First, in §4.3.1, we provide a summary of some phases or phenomena newly predicted by our computations which might be interesting to investigate further. In §4.3.2, we introduce and review the tools from algebraic topology we need to make these computations: the Adams and Atiyah-Hirzebruch spectral sequences. In §4.3.3, we discuss how to use the Adams filtration to detect when an invertible TFT of $\widetilde{H}$-manifolds only depends on the underlying $\mathrm{SO} \times G$-structure, which is believed to correspond to detecting which fermionic phases are really bosonic phases that are fermionic in a trivial way. Finally, in §4.3.4, we state and prove several lemmas needed in the computations in the next sections.

Then, in §§4.4–4.5, we implement this in examples, computing phase homology groups of $\mathbb{R}^d$ equivariant for two- and three-dimensional point-group symmetries, which in Ansatz 4.1.19 are interpreted as groups of point group equivariant fermionic phases on $\mathbb{R}^d$. In all cases we consider Altland-Zirnbauer classes D and A (corresponding to symmetry types spin and spin^c, respectively), and consider phases with spinless fermions and spin-1/2 fermions. These computations amount to computing spin and spin^c bordism groups of Thom spectra of vector bundles over BG, where G is the point group of interest; we use the Adams and Atiyah-Hirzebruch spectral sequences to determine these bordism groups.

In §4.4, we consider $\mathbb{Z}/2$ acting by a reflection (§4.4.1) and by an inversion (§4.4.2), as well as C_n acting by rotations (§4.4.3) and D_{2n} acting by rotations and reflections on $\mathbb{R}^2$ (§4.4.4) or purely by rotations on $\mathbb{R}^3$ (§4.4.5). The results of these computations can be found in Tables 1, 2, 3, 4 and 5. Most of these symmetry types have been studied in the physics literature, and we compare our results with other researchers'.

In §4.5, we study many 3d point groups, including chiral tetrahedral symmetry (§4.5.1), pyritohedral symmetry (§4.5.2), full tetrahedral symmetry (§4.5.3), chiral octahedral symmetry (§4.5.4), full octahedral symmetry (§4.5.5), chiral icosahedral symmetry (§4.5.6), and full icosahedral symmetry (§4.5.7). In all cases, we study phases with spinless and spin-1/2 fermions in Altland-Zirnbauer types D and A. Our predictions in this section are new as far as we can determine. See Table 6 for the results of the computations.

In §4.6, we discuss phases equivariant for a glide reflection symmetry. Lu-Shi-Lu [**LSL17**] conjecture a general classification of such phases, and we translate their conjecture into a statement on phase homology groups using Freed-Hopkins' ansatz, then prove that statement. Finally, in §4.7, we suggest some directions for further research.

4.1. Phases on a G-space: the general principle

We reprise the ansatz of Freed-Hopkins [**FH19a**, Ansatzes 2.1, 3.3] on invertible phases on a G-space, though we need to generalize it: physicists often consider crystalline phases in which the symmetry acting on spacetime mixes with the internal symmetry (e.g. a reflection squaring to $(-1)^F$), leading us to generalize from homology to twisted homology.

What we do not do is define a phase of matter. Precisely defining topological phases of matter, even in the absence of spatial symmetries, is a difficult open question. Our ansatz is a heuristic that these objects can be classified with what we call *phase homology*, which we do define.

4.1.1. Invertible phases on a space. Let Y be a locally compact topological space and C an ∞-category.[3] Following Ando-Blumberg-Gepner [**ABG10, ABG18**], we say a C*-valued local system* on Y is a functor $\mathcal{L}\colon \pi_{\le\infty}Y \to \mathsf{C}$ here $\pi_{\le\infty}Y$ is the fundamental ∞-groupoid of Y.[4] If $\mathcal{L}\colon Y \to \mathsf{Sp}$ is a local system of spectra, the homology of Y valued in $\mathcal{L}$ is $\mathcal{L}_*(Y) \coloneqq \pi_*(\operatorname{hocolim}\mathcal{L})$, and the cohomology of Y valued in $\mathcal{L}$ is $\mathcal{L}^*(Y) \coloneqq \pi_*(\operatorname{holim}\mathcal{L})$; this generalizes (co)homology with local coefficients.

Given a subspace $j\colon Y' \hookrightarrow Y$, we also define relative homology groups: j induces a map

$$j_*\colon \operatorname*{hocolim}_{Y'} \mathcal{L}|_{Y'} \longrightarrow \operatorname*{hocolim}_{Y} \mathcal{L}, \tag{4.1.1}$$

and we define $\mathcal{L}(Y, Y') \coloneqq \pi_*(\operatorname{cofib}(j_*))$. Relative cohomology is analogous.

Definition 4.1.2. Assume that the one-point compactification $\overline{Y}$ of Y is a finite CW complex and $\mathcal{L}$ extends to a local system $\overline{\mathcal{L}}\colon \overline{Y} \to \mathsf{Sp}$. Choose such an extension $\overline{\mathcal{L}}$ over the basepoint $*$. The *Borel-Moore homology* of Y valued in $\mathcal{L}$ is

$$\mathcal{L}_{\mathrm{BM},*}(Y) \coloneqq \overline{\mathcal{L}}_*(\overline{Y}, *). \tag{4.1.3}$$

Definition 4.1.2 appears to depend on the choice of extension of $\mathcal{L}$ to $\overline{Y}$, but given two choices of extension, the cofibers of the induced maps $\operatorname{hocolim}\overline{\mathcal{L}}|_* \to \operatorname{hocolim}\overline{\mathcal{L}}$ are equivalent, hence compute the same Borel-Moore homology groups.

When $\mathcal{L}$ is constant, this recovers the usual notion of Borel-Moore (generalized) homology [**BM60, Mil95**].

Recall that a *symmetry type* is a space B with a map $f\colon B \to B\mathrm{O}$.

Definition 4.1.4. A *local system of symmetry types* over the space Y is a local system on Y valued in the ∞-category of spaces with a map to $B\mathrm{O}$.

This is closely related to Raptis-Steimle's definition of parametrized tangential structures [**RS17**, §2].

[3] There are different definitions of ∞-categories; we work with *quasicategories* as developed by Joyal [**Joy02**] and Lurie [**Lur09a**], so as to follow [**ABG10, ABG18**]. However, this paper does not depend on implementation-specific details. See [**ABG18**, §2] for more information and some useful references.

[4] This is not the only approach to parametrized homotopy theory; see also May-Sigurdsson [**MS06**] and Braunack-Mayer [**BM19**].

Symmetry types often arise as the stabilizations in n of maps $B\rho_n\colon BH_n \to B\mathrm{O}_n$ induced from representations $\rho_n\colon H_n \to \mathrm{O}_n$; see [**FH16a**, §2] for a general discussion. Likewise, the local systems of symmetry types we consider arise from BH-bundles over Y.

We repeatedly use the notion of *Thom spectra*; the definition given by Freed-Hopkins [**FH16a**, §6.1.4] covers the cases we need.[5]

Let $\mathrm{Th}\colon \mathsf{Top}_{/B\mathrm{O}} \to \mathsf{Sp}$ denote the Thom spectrum functor and $\mathcal{I}\colon \mathsf{Sp}^{\mathrm{op}} \to \mathsf{Sp}$ denote the functor $\mathrm{Map}(-, \Sigma^2 I_{\mathbb{Z}})$.

Definition 4.1.5. Let Y be a locally compact space and $f\colon Y \to \mathsf{Top}_{/B\mathrm{O}}$ be a parametrized symmetry type on Y. The *phase homology* of this data, denoted $Ph_*(Y; f)$, is the Borel-Moore homology

$$Ph_*(Y; f) \coloneqq (\mathcal{I} \circ \mathrm{Th} \circ f)_{\mathrm{BM},*}(Y). \tag{4.1.6}$$

Ansatz 4.1.7. With Y and f as in Definition 4.1.5, the group of invertible topological phases on Y for the local system of symmetry types f is the phase homology group $Ph_0(Y; f)$.

Again, this is not a mathematical definition, but rather a heuristic.

Remark 4.1.8. When f is constant, Ansatz 4.1.7 is the original ansatz of Freed-Hopkins [**FH19a**, Ansatz 2.1]. In that case, the ansatz builds on the idea that invertible phases on Y are related to families of reflection positive invertible field theories on Y. The generalization to nonconstant f allows one to prescribe how the symmetry type of the family varies along Y. For example, one might want to consider families of phases in which the monodromy around a loop in Y acts by orientation reversal.

4.1.2. Invertible phases on a G-space. Our model for invertible crystalline phases requires considering the case where a compact Lie group G acts on Y. Again we closely follow Freed-Hopkins [**FH19a**, §3] but using twisted Borel-Moore homology.

Throughout this section, G is a Lie group; unlike in [**FH16a, FH19a**], we do not need G to be compact. Indeed, in the study of crystalline phases, G is often an infinite discrete subgroup of $\mathrm{Isom}(\mathbb{E}^n)$, and we will consider one such example in §4.6. We work with the ∞-category Sp^G of *Borel G-equivariant spectra*, whose objects can be modeled by data of a sequence of G-spaces X_n together with G-equivariant maps

[5] Thom spectra have been heavily studied in homotopy theory; key references include Thom [**Tho54**], Atiyah [**Ati61b**], May-Quinn-Ray-Tornehave [**May77**], and Ando-Blumberg-Gepner-Hopkins-Rezk [**ABG$^+$14a, ABG$^+$14b**].

$\Sigma X_n \to X_{n+1}$.[6] Notions of homotopy equivalence, etc., are as in [**FH16a**, 6.1], and do not require their compactness assumption on G.

Definition 4.1.9. Suppose G admits a finite-dimensional, real orthogonal representation $\lambda\colon G \to \mathrm{O}_d$. The one-point compactification of $\mathbb{R}^d$ with this G-action is a G-space denoted S^λ and called a *representation sphere.*

The suspension functor $\Sigma^\lambda \coloneqq S^\lambda \wedge -$ is not invertible in G-spaces, but upon stabilization is invertible in Borel G-spectra; we denote its inverse by $\Sigma^{-\lambda}$. Given a virtual G-representation $V = \lambda - \mu$ (i.e. a formal difference of two finite-dimensional real orthogonal representations), we define the Borel G-spectrum $\mathbb{S}^V \coloneqq \Sigma^{-\mu}\Sigma^\infty S^\lambda$. We will let $\mathbb{S}$ denote the sphere spectrum with trivial G-action.

Definition 4.1.10. Let Y be a G-space and $\mathcal{L}\colon Y \to \mathsf{Sp}^G$ be a local system. The *(Borel-)equivariant homology* of Y with respect to $\mathcal{L}$ is denoted $\mathcal{L}_*^G(Y)$ and defined to be

$$\mathcal{L}_*^G(Y) \coloneqq \pi_*(\mathrm{Map}_{\mathsf{Sp}^G}(\mathbb{S}, \operatorname{hocolim} \mathcal{L})^{hG}), \tag{4.1.11}$$

where $(-)^{hG}\colon \mathsf{Sp}^G \to \mathsf{Sp}$ denotes the homotopy fixed-points functor.

If $j\colon Y' \hookrightarrow Y$ is an inclusion of G-spaces, it induces a map

$$j_*\colon \mathrm{Map}_{\mathsf{Sp}^G}(\mathbb{S}, \operatorname*{hocolim}_{Y'} \mathcal{L}|_{Y'})^{hG} \longrightarrow \mathrm{Map}_{\mathsf{Sp}^G}(\mathbb{S}, \operatorname*{hocolim}_{Y} \mathcal{L})^{hG}, \tag{4.1.12}$$

and we define the *relative (Borel-)equivariant homology*

$$\mathcal{L}_*^G(Y, Y') \coloneqq \pi_*(\mathrm{cofib}(j_*)) \tag{4.1.13}$$

as in the nonequivariant case.

Definition 4.1.14. Let Y be a G-space and $\mathcal{L}\colon Y \to \mathsf{Sp}^G$ be an Sp^G-valued local system. Assume that the one-point compactification $\overline{Y}$ of Y is a CW complex and $\mathcal{L}$ extends to a local system $\overline{\mathcal{L}}\colon \overline{Y} \to \mathsf{Sp}^G$. Choose such an extension $\overline{\mathcal{L}}$. The *equivariant Borel-Moore homology* of Y valued in $\mathcal{L}$ is

$$\mathcal{L}_{\mathrm{BM},*}^G(Y) \coloneqq \overline{\mathcal{L}}_*^G(\overline{Y}, *). \tag{4.1.15}$$

Just like Definition 4.1.2, this does not actually depend on the choice of extension.

[6] There are a few different notions of G-spectra in the equivariant homotopy theory literature, and their names can be confusing. Borel G-equivariant spectra can be thought of as "spectra with a G-action" or "spectra living over BG," and are different from *genuine G-spectra*, which have a richer structure. To a geometer, "equivariant (generalized) cohomology" usually means the Borel theory, but to a homotopy theorist, it means the genuine theory. See [**Sul20**, §2.1] for a detailed introduction into the different names and notions of G-spaces and G-spectra.

Definition 4.1.16. Let Y be a G-space. A *G-equivariant local system of symmetry types* is a G-space B and a G-equivariant map $f\colon B \to Y \times B\mathrm{O}$, where $B\mathrm{O}$ has a trivial G-action.

Taking the Thom spectrum of the map to $B\mathrm{O}$ defines a local system $\mathrm{Th} \circ f\colon Y \to \mathsf{Sp}^G$.

Definition 4.1.17. Let Y be a G-space whose one-point compactification is a finite CW complex, and let $f\colon B \to Y \times B\mathrm{O}$ be a G-equivariant local system of symmetry types for Y. The *G-equivariant phase homology* of this data, denoted $Ph_*^G(Y; f)$, is the equivariant Borel-Moore homology

$$Ph_*^G(Y; f) := (\mathcal{I} \circ \mathrm{Th} \circ f)^G_{\mathrm{BM},0}(Y). \tag{4.1.18}$$

Ansatz 4.1.19. With Y and f as in Definition 4.1.17, the group of invertible topological phases on Y for the equivariant local system of symmetry types f is the G-equivariant phase homology group $Ph_0^G(Y; f)$.

Again, this is a heuristic and not a definition. When G is a discrete subgroup of $\mathrm{Isom}(\mathbb{E}^n)$ (e.g. a wallpaper or space group) acting on $Y = \mathbb{E}^n$, these phases are called *crystalline SPT phases* in the physics literature.

4.1.3. Mixing internal and crystalline symmetries. The fermionic crystalline equivalence principle is about invertible topological phases in which an internal symmetry mixes with the symmetry group acting on space. In this section, we construct the equivariant local systems of symmetry types for these phases. First, we review how mixing of symmetries is handled in the purely internal case in Example 4.1.20; then we address the case of spatial symmetries in Proposition 4.1.29, showing how to reduce the computation of the relevant equivariant phase homology groups to a nonparametrized question. We will simplify these computations further in §4.2 when we discuss the FCEP in more detail, then study several examples in §§4.4–4.5.

Example 4.1.20 (Mixing for internal symmetries)**.** In the study of SPTs, one commonly encounters symmetry types where there are two different symmetries present, such as time reversal and fermion parity, but they mix, meaning the group they generate is not a product of the individual symmetry groups, but rather an extension. For example, we could ask for a generator T of the group of time-reversal symmetries to square to the fermion parity $(-1)^F$, via the extension $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$, rather than considering phases where $T^2 = 1$, corresponding to the split extension $0 \to \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$.

Freed-Hopkins [**FH16a**] make the ansatz that SPT phases are classified up to equivalence by their low-energy limits, which are invertible field theories. The symmetry type is expressed as an H_n-structure, where H_n is a group with a map to O_n; mixing manifests as an extension involving the base symmetry type (e.g. Spin_n for fermionic phases) and the additional symmetry. For example, the two cases of time-reversal symmetry squaring to the identity or to fermion parity are represented by the extensions

$$1 \longrightarrow \mathrm{Spin}_n \longrightarrow \mathrm{Pin}_n^+ \longrightarrow \mathbb{Z}/2 \longrightarrow 1 \tag{4.1.21a}$$

$$1 \longrightarrow \mathrm{Spin}_n \longrightarrow \mathrm{Pin}_n^- \longrightarrow \mathbb{Z}/2 \longrightarrow 1, \tag{4.1.21b}$$

respectively, together with the standard maps $\mathrm{Pin}_n^\pm \to \mathrm{O}_n$.

When one of the groups we want to mix acts on space, we can specify a mixed symmetry type by the following data:

- a symmetry type $\rho_n\colon H_n \to \mathrm{O}_n$, called the *base symmetry type*,
- the point group symmetry $\lambda\colon G \to \mathrm{O}_d$,
- an extension

$$1 \longrightarrow H_n \longrightarrow \widetilde{H}_n \longrightarrow G \longrightarrow 1 \tag{4.1.22}$$

  specifying how they mix, and
- an extension $\widetilde{\rho}_n\colon \widetilde{H}_n \to \mathrm{O}_n$ of $\rho_n\colon H_n \to \mathrm{O}_n$.

Freed-Hopkins [**FH16a**, §9.2] relate Altland-Zirnbauer's symmetry classes of condensed-matter systems [**Zir96, AZ97**] to ten symmetry types in topology.[7] Using this, we call the case $H = \mathrm{Spin}$ the *class D case* and $H = \mathrm{Spin}^c$ the *class A case*.

Let Y be a G-space. Then the map

$$Y \times E\widetilde{H}_n/H_n \longrightarrow Y \tag{4.1.23}$$

is a G-equivariant fiber bundle with fiber BH_n, and the total space maps to $B\mathrm{O}_n$ as specified by the virtual vector bundle

$$f\colon -(Y \times (E\widetilde{H}_n \times_{H_n} \mathbb{R}^n)) \longrightarrow Y \times E\widetilde{H}_n/H_n. \tag{4.1.24}$$

After stabilizing (i.e. letting $n \to \infty$), this is an equivariant local system of symmetry types over Y, so has equivariant phase homology groups $Ph_*^G(Y; f)$. Under Ansatz 4.1.19, $Ph_0^G(Y; f)$ models the group of invertible topological phases on Y in which fermion parity mixes with the spatial symmetry as specified

[7] This "tenfold way" is a relativistic version of Dyson's threefold way [**Dys62**], and appears in many contexts in physics, including [**Kit09, RSFL10, FM13, WS14, FH16a, KZ16, GM20, IT20**].

by (4.1.22). The notion of G-equivariant phases for this symmetry type (without a reference space Y) is taken to mean G-equivariant phases on $\mathbb{R}^d$, where G acts on $\mathbb{R}^d$ through λ.

Remark 4.1.25 (Change of symmetry type)**.** We would like to be able to move information between instances of this data: for example, there should be forgetful maps from equivariant phases on a space to nonequivariant ones, and we model them with maps between phase homology groups for the two local systems of symmetry types.

Suppose we are given two instances of the data above. That is, we ask for a commutative diagram of Lie groups

$$\begin{tikzcd} 1 \arrow[r] & H_n \arrow[r] \arrow[d, "\varphi"] & \widetilde{H}_n \arrow[r] \arrow[d, "\widetilde{\varphi}"] & G \arrow[r] \arrow[d, "\varphi_G"] & 1 \\ 1 \arrow[r] & H'_n \arrow[r] & \widetilde{H}'_n \arrow[r] & G' \arrow[r] & 1 \end{tikzcd} \tag{4.1.26}$$

together with maps $\rho_n\colon H_n \to \mathrm{O}_n$ and $\rho'_n\colon H_n \to \mathrm{O}_n$, $\lambda\colon G \to \mathrm{O}_d$ and $\lambda'\colon G' \to \mathrm{O}_d$, and $\widetilde{\rho}_n\colon \widetilde{H}_n \to \mathrm{O}_n$ and $\widetilde{\rho}'_n\colon \widetilde{H}'_n \to \mathrm{O}_n$ which commute with the vertical maps in (4.1.26). Fix a G'-space Y; then through (4.1.24) this defines equivariant local systems of symmetry types f for G, resp. f' for G'. The maps between the data induce a pullback or forgetful map $\varphi^*\colon Ph_*^{G'}(Y; f') \to Ph_*^G(Y; f)$, where G acts on Y through φ_G. Using Ansatz 4.1.19, we interpret this pullback map realizing an invertible phase on Y with a G'-symmetry to a phase with a G-symmetry.

The construction of φ^* amounts to checking that diagrams you would expect to commute do in fact commute. The data we gave induces a commutative diagram

$$\begin{tikzcd} -(Y \times (E\widetilde{H}_n \times_{H_n} \mathbb{R}^n)) \arrow[r] \arrow[d] & Y \times E\widetilde{H}_n/H_n \arrow[d] \\ -(Y \times (E\widetilde{H}'_n \times_{H'_n} \mathbb{R}^n)) \arrow[r] & Y \times E\widetilde{H}'_n/H'_n. \end{tikzcd} \tag{4.1.27}$$

The rows define equivariant local systems symmetry types; then f and f' are the maps to $Y \times B\mathrm{O}$. Let $\varphi^\circ\colon \mathsf{Sp}^{G'} \to \mathsf{Sp}^G$ be the map in which G acts on Borel G'-spectra through φ; then, upon applying $\mathcal{I} \circ \mathrm{Th}$, we obtain local systems $\mathcal{L}$, resp. $\mathcal{L}'$ of Borel G-, resp. G'-spectra. To define phase homology, we assumed that an extension $\overline{\mathcal{L}}$ of $\mathcal{L}$ to $\overline{Y}$ exists, so choose such an extension; then $\overline{\mathcal{L}}' := \overline{\mathcal{L}} \circ \varphi^\circ$ is an extension of $\mathcal{L}$. We obtain from the inclusion $* \hookrightarrow \overline{Y}$ a commutative diagram of spectra

$$\begin{tikzcd} \mathrm{Map}_{\mathsf{Sp}^{G'}}(\mathbb{S}, \operatorname*{hocolim}_{*} \overline{\mathcal{L}}'|_*)^{hG'} \arrow[r] \arrow[d] & \mathrm{Map}_{\mathsf{Sp}^{G}}(\mathbb{S}, \operatorname*{hocolim}_{*} \overline{\mathcal{L}}|_*)^{hG} \arrow[d] \\ \mathrm{Map}_{\mathsf{Sp}^{G'}}(\mathbb{S}, \operatorname*{hocolim}_{\overline{Y}} \overline{\mathcal{L}}')^{hG'} \arrow[r] & \mathrm{Map}_{\mathsf{Sp}^{G}}(\mathbb{S}, \operatorname*{hocolim}_{\overline{Y}} \overline{\mathcal{L}})^{hG}. \end{tikzcd} \tag{4.1.28}$$

Thus, we get a map between the cofibers of the vertical arrows, and π_* of that map is the desired map on phase homology.

For us there are two particularly important examples.

(1) Let $H_n' = H_n$ and $G = 1$, which forces $\widetilde{\varphi}\colon \widetilde{H}_n \to \widetilde{H}_n'$ to be the inclusion $H_n' \to \widetilde{H}_n'$. The above construction produces a map from H-equivariant phase homology to nonequivariant phase homology on Y, which we interpret as modeling the forgetful map from phases with a G-symmetry to phases without a G-symmetry.

(2) Let $G' = G$, $H_n' = \mathrm{SO}_n$. and H_n be either Spin_n or Spin_n^c, with φ the usual map. In this case the pullback map goes from equivariant phase homology where the base symmetry type is SO to equivariant phase homology where the base symmetry type is Spin or Spin^c. We interpret this in physics as modeling the procedure that regards a bosonic phase as a fermionic phase by adding some trivial fermionic degrees of freedom. This is analogous to the procedure which regards an oriented TFT as a spin TFT that does not depend on the spin structure.

Crucially for computations, we can simplify the equivariant phase homology groups for the symmetry types in (4.1.24) into a description not requiring equivariant or parametrized stable homotopy theory.

Proposition 4.1.29. *There is an isomorphism*

$$Ph_0^G(\mathbb{R}^d; f) \xrightarrow{\cong} [(B\widetilde{H})^{d-\lambda-\widetilde{\rho}}, \Sigma^{d+2} I_{\mathbb{Z}}] \tag{4.1.30}$$

natural for changing the symmetry type in the sense of Remark 4.1.25.

Proof. We want to compute the twisted equivariant Borel-Moore homology for this equivariant local system of symmetry types, where $Y = \mathbb{R}^d$ with G acting through λ. This amounts to the following: one-point compactify to a local system over S^λ; take the colimit of the local system and call it E; then compute $[\mathbb{S}, E]^G$ (in the notation of [**FH19a**]; this means $\pi_0(\mathrm{Map}(\mathbb{S}, E)^{hG})$). Now, the local system $(\mathcal{I} \circ \mathrm{Th} \circ f)\colon S^\lambda \to \mathsf{Sp}^G$ is nonequivariantly the trivial local system with fiber $\mathrm{Map}(MTH, \Sigma^2 I_{\mathbb{Z}})$, so $E \simeq S^\lambda \wedge \mathrm{Map}(MTH, \Sigma^2 I_{\mathbb{Z}})$; in general, G can act nontrivially on both S^λ and MTH, but always acts trivially on $\Sigma^2 I_{\mathbb{Z}}$. Therefore we may follow [**FH19a**, (3.6)] and identify

$$\mathrm{Map}(\mathbb{S}, S^\lambda \wedge \mathrm{Map}(MTH, \Sigma^2 I_{\mathbb{Z}})) \simeq \mathrm{Map}(\mathbb{S}^{d-\lambda} \wedge MTH, \Sigma^{d+2} I_{\mathbb{Z}}), \tag{4.1.31}$$

though the G-action on $\mathbb{S}^{d-\lambda} \wedge MTH$ is not the diagonal action, but rather the induced G-action on the Thom spectrum of the G-equivariant virtual bundle $(d - \underline{\lambda} - \rho) \to BH$ (see [**FH16a**, §6.2.2]).

Since G acts trivially on $\Sigma^{d+2} I_{\mathbb{Z}}$,

$$\mathrm{Map}(\mathbb{S}^{d-\lambda} \wedge MTH, \Sigma^{d+2} I_{\mathbb{Z}})^{hG} \simeq \mathrm{Map}((\mathbb{S}^{d-\lambda} \wedge MTH)_{hG}, \Sigma^{d+2} I_{\mathbb{Z}}). \tag{4.1.32}$$

It now suffices to show that

$$(\mathbb{S}^{d-\lambda} \wedge MTH)_{hG} \simeq (B\widetilde{H})^{-\widetilde{\rho}-\lambda+d}. \tag{4.1.33}$$

Ando-Blumberg-Gepner-Hopkins-Rezk [**ABG**$^+$**14a**, Proposition 1.20] show that the Thom spectrum of a virtual bundle $V \to X$, identified with a map $V\colon X \to B\mathrm{O}$, is the homotopy colimit

$$X^V \simeq \mathrm{hocolim}\big(X \xrightarrow{V} B\mathrm{O} \xrightarrow{BJ} B\mathrm{GL}_1(\mathbb{S}) \longrightarrow \mathsf{Sp}\big), \tag{4.1.34}$$

where the notation means to interpret X as, through its fundamental ∞-groupoid, providing a diagram in the ∞-category Sp of spectra. Here $B\mathrm{GL}_1(\mathbb{S})$ is the classifying space of stable spherical fibrations [**Sta63, May77**] and $BJ\colon B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S})$ is a form of the J-homomorphism [**Whi42, May77**]. Heuristically, (4.1.34) says that the virtual vector bundle V defines a local system of $\wedge$-invertible spectra, with the fiber at a point $x \in X$ given by $\mathbb{S}^{V_x}$, and that the Thom spectrum is obtained from an associated bundle construction. See [**ABG**$^+$**14a, ABG**$^+$**14b**] for more detail on this approach to Thom spectra.

Homotopy quotients are also homotopy colimits, meaning

$$(\mathbb{S}^{d-\lambda} \wedge MTH)_{hG} = \underset{\mathrm{pt}/G}{\mathrm{hocolim}}\Big(\mathrm{hocolim}\big(BH \xrightarrow{d-\underline{\lambda}-\rho} B\mathrm{O} \xrightarrow{BJ} B\mathrm{GL}_1(\mathbb{S}) \longrightarrow \mathsf{Sp}\big)\Big), \tag{4.1.35a}$$

where G acts on the spectra in the diagram through its action on λ, as well as on BH, as prescribed by the extension (4.1.22). This in particular implies the double homotopy colimit above simplifies into a single homotopy colimit over a $B\widetilde{H}$-shaped diagram:

$$\simeq \mathrm{hocolim}\Big(B\widetilde{H} \xrightarrow{d-\lambda-\widetilde{\rho}} B\mathrm{O} \xrightarrow{BJ} B\mathrm{GL}_1(\mathbb{S}) \longrightarrow \mathsf{Sp}\Big), \tag{4.1.35b}$$

which by (4.1.34) is the Thom spectrum for $d - \lambda - \widetilde{\rho} \to B\widetilde{H}$, proving (4.1.33). $\square$

Our next step in §4.2 is to simplify $(B\widetilde{H})^{d-\lambda-\widetilde{\rho}}$. This allows both for a general formulation of the fermionic crystalline equivalence principle as well as explicit calculations.

The following lemma will be helpful for simplifying Thom spectra.

Theorem 4.1.36 (Relative Thom isomorphism)**.** *Let $\rho\colon H \to \mathrm{O}$ be a symmetry type with the two-out-of-three property, i.e. an H-structure on any two of E, F, or $E \oplus F$ induces one on the third. If $V, W \to X$ are virtual vector bundles such that V has an H-structure, then there is an equivalence*

$$MTH \wedge X^W \xrightarrow{\simeq} MTH \wedge X^{V \oplus W}. \tag{4.1.37}$$

Proof. The two-out-of-three property gives MTH an E_∞-ring structure, which is needed for some of the constructions we employ from [**ABG+14a, ABG+14b**] below.

Up to equivalence, the Thom spectrum of a virtual vector bundle $E \to X$ depends only on the homotopy class of the map $f_E\colon X \to B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S})$, where the first map is given by E, and the second map is the J-homomorphism, as in (4.1.34). Smashing with MTH corresponds to composing f_E with the map $B\mathrm{GL}_1(\mathbb{S}) \to B\mathrm{GL}_1(MTH)$ induced by the Hurewicz map $\mathbb{S} \to MTH$ [**ABG+14b**, §1.4], and in particular, up to equivalence, $MTH \wedge X^E$ only depends on the homotopy type of the map $X \to B\mathrm{GL}_1(MTH)$.

Because MTH is an E_∞-ring spectrum, $B\mathrm{GL}_1(MTH)$ is a grouplike E_∞-space, and the composition $\psi\colon B\mathrm{O} \to B\mathrm{GL}_1(\mathbb{S}) \to B\mathrm{GL}_1(MTH)$ is a map of grouplike E_∞-spaces, where $B\mathrm{O}$ carries the E_∞ structure coming from direct sum. This means that $[X, B\mathrm{GL}_1(MTH)]$ is naturally an abelian group, and that if we define classes in this group using virtual vector bundles $V, W \to X$ to map to $B\mathrm{O}$ then composing with ψ, the class of $E \oplus F$ is the sum of the classes of V and W.

An H-structure on V trivializes the map $X \to B\mathrm{O} \xrightarrow{\psi} B\mathrm{GL}_1(MTH)$ defined by V, so the class of the map defined by $V \oplus W$ is equal to the class of the map defined by W in the abelian group $[X, B\mathrm{GL}_1(MTH)]$. $\square$

4.2. The fermionic crystalline equivalence principle

In this section, our goal is to state and prove the FCEP, Theorem 4.2.8, identifying phase homology groups in classes D and A with groups of deformation classes of invertible field theories. Assuming Ansatz 4.1.19, this leads to the more familiar version of the FCEP: crystalline equivalence principles are first introduced by Thorngren-Else [**TE18**]: the idea is to equate the classification of crystalline topological phases of matter for some group G acting on spacetime with a classification of a different kind of topological phases of matter, in which G is part of the internal symmetry group. Then one may use preexisting techniques for phases without a spatial symmetry to classify phases with the specified G-action on space.

The best-understood crystalline equivalence principles are for bosonic SPTs, as first considered by Thorngren-Else [**TE18**]. "Bosonic" does not have a precise mathematical translation here; these are phases for which the symmetry type is built using SO or O rather than Spin, Spin^c, Pin$^\pm$, and so on. If a group G acts on space by orientation-preserving symmetries and H is SO or O, the classification of crystalline SPTs in dimension n with symmetry type H and this G-action is identified with the classification of SPTs for $H = \mathrm{SO} \times G$. To what extent this is an ansatz or a theorem depends on one's model for crystalline SPTs: Freed-Hopkins [**FH19a**, Example 3.5] derive it as a corollary of their ansatz.[8] For other derivations of the bosonic crystalline equivalence principle from different ansatzes, see Jiang-Ran [**JR17**] and Thorngren-Else [**TE18, ET19**].

The fermionic analogue of this statement is more complicated because there are more ways for G to mix with the symmetry type. Thorngren-Else [**TE18**, §VII.B], Cheng-Wang [**CW18**], Zhang-Wang-Yang-Qi-Gu [**ZWY$^+$20**], and Zhang-Wang-Yang-Gu [**ZYQG20**, §V] all study examples in which an FCEP holds, and each paper discusses that such a principle would have to account for the different ways in which G mixes with H: crystalline phases for which the spatial G-symmetry does not mix with fermion parity correspond to phases with an internal G-symmetry that does mix with fermion parity, and vice versa. Examples of this twisted correspondence also appear in work of Freed-Hopkins [**FH19a**, Example 3.5], Guo-Ohmori-Putrov-Wan-Wang [**GOP$^+$20**], and Mao-Wang [**MW20**], though until now there was no precise general version of the FCEP.

Our version of the FCEP applies in Altland-Zirnbauer classes A and D (i.e. $H = \mathrm{Spin}$ or $H = \mathrm{Spin}^c$), for all compact Lie groups G acting on faithfully on space, and all ways in which G may mix with fermion parity. The slogan "mixed crystalline goes to unmixed internal, and vice versa" is a little hard to glean from the result when the G-action includes reflections, but we obtain an equivalence from phase homology groups for certain equivariant local systems of symmetry types, which under Ansatz 4.1.19 stands in for groups of crystalline SPT phases, to groups of deformation classes of IFTs, which under Freed-Hopkins' ansatz [**FH16a**] model groups of phases without spatial symmetries.

To precisely state our FCEP, we must fix some data.

Data 4.2.1.

- Let H denote the base symmetry type, which today is either of the infinite-dimensional topological groups Spin or Spin^c.

[8] If G acts by reflections, almost as nice of a story is still true, but the internal G-symmetry mixes with H. Thorngren-Else [**TE18**] and Freed-Hopkins [**FH19a**, Example 3.5] discuss this case too.

- Let G be a compact Lie group, $\lambda\colon G \to \mathrm{O}_d$ be a faithful representation, and $V_\lambda \coloneqq EG \times_G \mathbb{R}^d \to BG$ be the associated vector bundle.
- Let $\xi\colon G \to \mathrm{O}_{d'}$ be another faithful representation and $V_\xi \to BG$ be the associated vector bundle. Let $1 \to \mu_2 \to \widetilde{G} \to G \to 1$ be the central extension classified by $w_2(V_\lambda) + w_1(V_\lambda)^2 \in H^2(BG;\mu_2)$. Here μ_2 denotes the group of square roots of unity.
- Let $\widetilde{H} \coloneqq H \times_{\mu_2} \widetilde{G}$. Let ρ be the composition $\widetilde{H} \to H \to \mathrm{O}$ and $V \to B\widetilde{H}$ be the associated tautological vector bundle.

For us, ξ and λ are usually the same, but they differ when $G = \mathbb{Z}/2$ acts on $\mathbb{R}^d$ by inversion in the case of spin-1/2 fermions: here ξ is the sign representation $\sigma\colon \mathbb{Z}/2 \to \mathrm{O}_1$, but $\lambda = d\sigma$. See §4.4.2 for more detail.

Definition 4.2.2. The *spin-*1/2 *equivariant local system of symmetry types* for the above data is the G-equivariant parametrized symmetry type $f_{1/2}\colon B\widetilde{H} \to \mathbb{R}^{d'} \times B\mathrm{O}$ which sends $x \mapsto (0, B\rho(x))$, and in which G acts on $\mathbb{R}^d$ through λ. The *spinless equivariant local system of symmetry types* f_0 is defined in the same way, except using $H \times G$ instead of $\widetilde{H}$.

Definition 4.2.3. Recall that H is either Spin or Spin^c. Let $\dagger \in \{-, c\}$ be $-$ if $H = \mathrm{Spin}$ and c otherwise. The *spinless internal symmetry type* is the symmetry type

- $(-V, d - V_\lambda)\colon BH \times BG \to B\mathrm{O}$, if λ is $\mathrm{pin}^\dagger$, or
- $(-V, V_\xi + \mathrm{Det}(V_\xi) - V_\lambda)\colon BH \times BG \to B\mathrm{O}$, if λ is not $\mathrm{pin}^\dagger$.

For shorthand, we denote this symmetry type $\rho(0)\colon BH \times BG \to B\mathrm{O}$.

Definition 4.2.4. The *spin-*1/2 *internal symmetry type* is the symmetry type

$$(-V, d - V_\lambda)\colon BH \times BG \to B\mathrm{O}. \tag{4.2.5}$$

For shorthand, we denote this symmetry type $\rho(1/2)\colon BH \times BG \to B\mathrm{O}$.

Remark 4.2.6. The internal symmetry types probably look pretty arbitrary. This is because of the generality of our setup: in some cases of interest, we can rewrite these symmetry types in ways which more closely resembles the proposals of Thorngren-Else [**TE18**, §VII.B], Cheng-Wang [**CW18**], and Zhang-Wang-Yang-Qi-Gu [**ZWY+20**] for the FCEP in specific cases.

Suppose $\lambda = \xi$ and $\mathrm{Im}(\lambda) \subset \mathrm{SO}_d$ but does not lift across $\mathrm{Spin}_d \twoheadrightarrow \mathrm{SO}_d$. Then, the spinless internal symmetry type simplifies to $BH \times BG \to B\mathrm{O}$, where the map is just projection onto the first factor followed by the usual map $BH \to B\mathrm{O}$. That is, for representations with image contained in SO_d, the FCEP switches the "unmixed" (i.e. $BH \times BG$) and "mixed" (i.e. $B(H \times_{\mu_2} \widetilde{G})$) symmetry types when passing between crystalline and internal phases. This matches predictions by Thorngren-Else [**TE18**] and Cheng-Wang [**CW18**].

Freed-Hopkins [**FH16a**, Corollary 8.21] show that the group of deformation classes of reflection positive IFTs with symmetry type $\rho'\colon H' \to \mathrm{O}$ in (space) dimension n is naturally isomorphic to[9]

$$[MTH', \Sigma^{d+2} I_{\mathbb{Z}}]. \tag{4.2.7}$$

Theorem 4.2.8 (Fermionic crystalline equivalence principle)**.** *There are isomorphisms*

$$Ph_k^G(\mathbb{R}^d; f_0) \xrightarrow{\cong} [MT\rho(1/2), \Sigma^{d+k+2} I_{\mathbb{Z}}] \tag{4.2.9a}$$

$$Ph_k^G(\mathbb{R}^d; f_{1/2}) \xrightarrow{\cong} [MT\rho(0), \Sigma^{d+k+2} I_{\mathbb{Z}}]. \tag{4.2.9b}$$

Assuming Ansatz 4.1.19, the physics implication of this theorem is that the abelian group of crystalline SPT phases for the spinless equivariant local system of symmetry types is naturally isomorphic to the abelian group of deformation classes of IFTs for the spin-1/2 internal symmetry type; and the classification of crystalline SPT phases for the spin-1/2 equivariant local system of symmetry types is naturally isomorphic to the abelian group of deformation classes of IFTs of the spinless internal symmetry type.

We break the proof of Theorem 4.2.8 down into a few steps. First, Proposition 4.1.29 simplifies the question into one of ordinary stable homotopy theory.[10] We obtain Thom spectra for vector bundles over $B\widetilde{H}$, and to finish we must compare these spectra to $MTH \wedge (BG)^E$, where $E \to BG$ is some rank-zero virtual vector bundle. This comparison, in the form of *shearing arguments*, is the core of the proof: we prove Theorem 4.2.11 ($H = \mathrm{Spin}$) and Theorem 4.2.24 ($H = \mathrm{Spin}^c$) establishing the homotopy equivalences we need, and after that proving Theorem 4.2.8 amounts to verifying that the outputs of Theorems 4.2.11 and 4.2.24 simplifying the crystalline symmetry types match the Thom spectra for the internal symmetry types in Definitions 4.2.3 and 4.2.4.

[9]Strictly speaking, Freed-Hopkins' theorem classifies only the invertible *topological* field theories, which form the torsion subgroup of (4.2.7), and they conjecture that the entire group classifies all reflection positive IFTs.
[10]For the spinless equivariant symmetry type, this is just [**FH19a**, Example 3.5].

The proofs of Theorems 4.2.11 and 4.2.24 resemble the proofs of the more standard equivalences

(4.2.10a) $$MTPin^+ \simeq MTSpin \wedge (B\mathbb{Z}/2)^{1-\sigma}$$

(4.2.10b) $$MTPin^- \simeq MTSpin \wedge (B\mathbb{Z}/2)^{\sigma-1}$$

(4.2.10c) $$MTPin^c \simeq MTSpin^c \wedge (B\mathbb{Z}/2)^{\pm(1-\sigma)}$$

(4.2.10d) $$MTSpin^c \simeq MTSpin \wedge (B\mathrm{SO}_2)^{\pm(2-V_2)},$$

where $\sigma \to B\mathbb{Z}/2$ and $V_2 \to B\mathrm{SO}_2$ denote the respective tautological line bundles. These decompositions were first proven by Kirby-Taylor [**KT90a**, Lemma 6] (pin$^+$), Peterson [**Pet68**, §7] (pin$^-$), and Bahri-Gilkey [**BG87a, BG87b**] (spin^c and pin^c). For a unified proof of all of these equivalences, see Freed-Hopkins [**FH16a**, §10].

4.2.1. Case $H = \mathrm{Spin}$.

Theorem 4.2.11 (Shearing, class D)**.** *Let $V \to B\widetilde{H}$ be the tautological bundle.*

(1) Suppose V_ξ admits a pin$^-$ structure. Then there is an equivalence

(4.2.12) $$(B\widetilde{H})^{d-V_\lambda-V} \xrightarrow{\simeq} MTSpin \wedge (BG)^{d-V_\lambda}.$$

(2) If V_ξ does not admit a pin$^-$ structure, there is an equivalence

(4.2.13) $$(B\widetilde{H})^{d-V_\lambda-V} \xrightarrow{\simeq} MTSpin \wedge (BG)^{V_\xi+\mathrm{Det}(V_\xi)-V_\lambda-d'-1+d}.$$

We will most often consider case (2) with $\lambda = \xi$, in which case we learn $(B\widetilde{H})^{d-\lambda-V} \simeq MTSpin \wedge (BG)^{\mathrm{Det}(V_\lambda)-1}$.

Proof. Case (1) is by far the easier of the two: V_ξ admits a pin$^-$ structure iff $w_2(V_\xi) + w_1(V_\xi)^2 = 0$ iff the extension $1 \to \mu_2 \to \widetilde{G} \to G \to 1$ splits. Since $\mu_2 \subset \widetilde{G}$ is central, a splitting induces isomorphisms $\widetilde{G} \cong \mu_2 \times G$ and $\widetilde{H}_n \cong \mathrm{Spin}_n \times G$. Passing to classifying spaces, this identifies $d - V_\lambda - V \colon B\widetilde{H} \to B\mathrm{O}$ with $-V \boxplus (d-\lambda) \colon B\mathrm{Spin} \times BG \to B\mathrm{O}$; then take Thom spectra.

On to case (2). In this case, in $H^2(B\widetilde{H};\mu_2)$, $w_2(V_\xi) + w_1(V_\xi)^2 = w_2(V)$, so the map $V + V_\xi + \mathrm{Det}(V_\xi) \colon B\widetilde{H} \to B\mathrm{SO}$ lifts across $B\mathrm{Spin} \to B\mathrm{SO}$. Choose such a lift.

Proposition 4.2.14. *The induced map*

(4.2.15) $$(V + V_\xi + \mathrm{Det}(V_\xi), \xi) \colon B\widetilde{H} \longrightarrow B\mathrm{Spin} \times BG$$

is a homotopy equivalence commuting with the maps to $B\mathrm{SO}$.

The proof is due to Freed-Hopkins [**FH16a**, §10].

PROOF. We will show that the commutative square

$$\begin{tikzcd} B\widetilde{H} \arrow[r] \arrow[d, "B(\pi_1\oplus\pi_2)"] & B\mathrm{Spin} \arrow[d] \\ B\mathrm{SO}\times BG \arrow[r, "B(\mathrm{id}\oplus\xi)"] & B\mathrm{SO} \end{tikzcd} \tag{4.2.16a}$$

is homotopy Cartesian. Any two homotopy pullbacks of the same diagram are weakly equivalent, with the weak equivalence intertwining the maps to $B\mathrm{SO}$. Since there is also a homotopy pullback square

$$\begin{tikzcd} B\mathrm{Spin}\times BG \arrow[r] \arrow[d] & B\mathrm{Spin} \arrow[d] \\ B\mathrm{SO}\times BG \arrow[r, "B(\mathrm{id}\oplus\xi)"] & B\mathrm{SO}, \end{tikzcd} \tag{4.2.16b}$$

then $B\widetilde{H} \simeq B\mathrm{Spin}\times BG$; this equivalence is realized by (4.2.15) because that is the only possibility that intertwines the maps in (4.2.16a) and (4.2.16b).

To fulfill the promise that (4.2.16a) is a homotopy pullback square, begin with the commutative diagram of short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, equal] & \widetilde{H}_n \arrow[r, "(\pi_1,\pi_2)"] \arrow[d] & \mathrm{SO}_n\times G \arrow[r] \arrow[d, "\mathrm{id}\oplus\xi"] & 1 \\ 1 \arrow[r] & \mu_2 \arrow[r] & \mathrm{Spin}_{n+d} \arrow[r] & \mathrm{SO}_{n+d} \arrow[r] & 1. \end{tikzcd} \tag{4.2.17}$$

This induces a map of fiber sequences

$$\begin{tikzcd} B\widetilde{H} \arrow[r, "B(\pi_1,\pi_2)"] \arrow[d] & B\mathrm{SO}\times BG \arrow[r, "w_2"] \arrow[d, "B(\mathrm{id}\oplus\xi)"] & K(\mu_2,2) \arrow[d, equal] \\ B\mathrm{Spin} \arrow[r] & B\mathrm{SO} \arrow[r, "w_2"] & K(\mu_2,2), \end{tikzcd} \tag{4.2.18}$$

e.g. $B\widetilde{H}$ is the fiber of $w_2\colon B\mathrm{SO}\times BG\to K(\mu_2,2)$. The left square in such a pullback is always homotopy Cartesian, and in (4.2.18) the left square can be identified with (4.2.16a). □

Including the maps down to $B\mathrm{SO}$ produces the commutative diagram

$$\begin{tikzcd} B\widetilde{H} \arrow[rr, "(V+V_\xi+\mathrm{Det}(V_\xi){,}\xi)", "\simeq"'] \arrow[dr, "-V"'] & & B\mathrm{Spin}\times BG \arrow[dl, "-V+V_\xi+\mathrm{Det}(V_\xi)"] \\ & B\mathrm{SO}. & \end{tikzcd} \tag{4.2.19}$$

Taking Thom spectra of the vertical maps, the shearing map induces a homotopy equivalence

$$(B\widetilde{H})^{-V} \xrightarrow{\simeq} \mathit{MTSpin}\wedge(BG)^{V_\xi+\mathrm{Det}(V_\xi)-d'-1}. \tag{4.2.20}$$

To finish, we subtract V_λ from the vertical arrows in (4.2.19), then take Thom spectra again. ☐

4.2.2. Case $H = \mathrm{Spin}^c$. Let $\widetilde{H}_n \coloneqq \mathrm{Spin}_n^c \times_{\mu_2} \widetilde{G}$, and define $\widetilde{H}$ similarly. The shearing argument is scarcely different than for Theorem 4.2.11, but it will be useful to rephrase $\widetilde{H}_n$ using the circle group $\mathbb{T}$ instead of μ_2.

The extension of G by μ_2 defines an extension of G by $\mathbb{T}$ by pushing forward along the inclusion $\mu_2 \hookrightarrow \mathbb{T}$:

$$\begin{tikzcd} 1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, hook] & \widetilde{G} \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathbb{T} \arrow[r] & \widehat{G} \arrow[r] & G \arrow[r] & 1. \end{tikzcd} \tag{4.2.21}$$

In cohomology, this construction is classified by the Bockstein map $H^2(BG;\mu_2)\to H^3(BG;\mathbb{Z})$. Let $\widehat{H}_n \coloneqq \mathrm{Spin}_n^c \times_{\mathbb{T}} \widehat{G}$ and $\widehat{H} \coloneqq \mathrm{Spin}^c \times_{\mathbb{T}} \widehat{G}$. The map $\widetilde{G}\to\widehat{G}$ induces maps $\varphi_n\colon \widetilde{H}_n \to \widehat{H}_n$ and $\varphi\colon \widetilde{H}\to\widehat{H}$; φ is the colimit of the φ_ns.

Lemma 4.2.22. *The maps $\varphi_n\colon \widetilde{H}_n \to \widehat{H}_n$ are isomorphisms of Lie groups.*

Proof. Write down the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, equal] & \widetilde{H}_n \arrow[r] \arrow[d, "\varphi"] & \mathrm{SO}_n\times\mathbb{T}\times G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mu_2 \arrow[r] & \widehat{H}_n \arrow[r] & \mathrm{SO}_n\times\mathbb{T}\times G \arrow[r] & 1 \end{tikzcd} \tag{4.2.23}$$

and apply the five lemma. ☐

And now we shear. Recall our notation from Data 4.2.1.

Theorem 4.2.24 (Shearing, class A)**.**

(1) Suppose V_ξ admits a pin^c structure. Then there is an equivalence

$$(B\widehat{H})^{d-V_\lambda-V} \xrightarrow{\simeq} MTSpin^c \wedge (BG)^{d-V_\lambda}. \tag{4.2.25}$$

(2) If V_ξ does not admit a pin^c structure, there is an equivalence

$$(B\widehat{H})^{d-V_\lambda-V} \xrightarrow{\simeq} MTSpin^c \wedge (BG)^{V_\xi+\mathrm{Det}(V_\xi)-V_\lambda-d'+1-d}. \tag{4.2.26}$$

Again, we most often use case (2) when $\lambda = \xi$, in which case the right-hand side simplifies to $MTSpin^c \wedge (BG)^{\mathrm{Det}(V_\lambda)-1}$.

Proof. The proof is barely different than that of Theorem 4.2.11; we indicate only the differences. In that theorem, the engine of the proof when V_ξ was not pin$^-$ was the map (4.2.15) from $B(\mathrm{Spin}\times_{\mu_2}\widetilde{G}) \to B\mathrm{Spin}\times BG$. Here, V_ξ is not pin^c, so $V_\xi \oplus \mathrm{Det}(V_\xi)$ is oriented but not spin^c. We have that if $\beta\colon H^2(B\widehat{H};\mu_2) \to H^3(B\widehat{H};\mathbb{Z})$ is the Bockstein, $\beta(w_2(V_\xi) + w_1(V_\xi)^2 + w_2(V)) = 0$, so $V + V_\xi + \mathrm{Det}(V_\xi)$, interpreted as a map $B\widehat{H} \to B\mathrm{SO}$, lifts to $B\mathrm{Spin}^c$. Our analogue of (4.2.15) is

$$(V + V_\xi + \mathrm{Det}(V_\xi), \xi)\colon B\widehat{H} \longrightarrow B\mathrm{Spin}^c \times BG. \tag{4.2.27}$$

As in Proposition 4.2.14, this is a homotopy equivalence commuting with the maps down to $B\mathrm{SO}$. The proof is almost the same, though we replace Spin with Spin^c in (4.2.16a) and (4.2.16b), μ_2 with $\mathbb{T}$ in (4.2.17), and $K(\mu_2, 2)$ with $K(\mathbb{Z}, 3)$ in (4.2.18). □

4.2.3. Putting it together. The hard work of the proof is already done.

Proof of Theorem 4.2.8. By Proposition 4.1.29,

$$Ph_0^G(\mathbb{R}^d; f_{1/2}) \cong [X, \Sigma^{d+1} I_\mathbb{Z}], \tag{4.2.28}$$

where $X := (B\widetilde{H})^{d-V_\lambda-V}$. Then Theorem 4.2.11 ($H = \mathrm{Spin}$) and Theorem 4.2.11 ($H = \mathrm{Spin}^c$) split this into $MTH \wedge (BG)^E$ for some rank-zero virtual vector bundle E. For f_0, because $\widetilde{H} \cong H \times G$, Proposition 4.1.29 gets us to $MTH \wedge (BG)^E$ without having to shear. The only thing left to do is compare these Thom spectra to Definitions 4.2.3 and 4.2.4, and sure enough, they match. □

4.3. Computations in examples: summary of results and some generalities

In the next two sections, we study the fermionic crystalline equivalence principle in many examples where the symmetry is given by a two- or three-dimensional point group. Here, we summarize the results and some takeaways for researchers interested in crystalline phases; for more detailed results of computations of groups of phases, see Tables 1, 2, 3, 4, 5 and 6.

In §4.3.1, we indicate some example phases predicted by our phase homology calculations that have not been previously studied to our knowledge, and which might have accessible or interesting lattice realizations. We also summarize which of our calculations correspond to phases already studied in the literature. In §4.3.2, we briefly review the computational techniques we use to study phase homology groups, namely the Adams and Atiyah-Hirzebruch spectral sequences. In §4.3.3, we use the Adams filtration to characterize which invertible field theories with $\widetilde{H}$-structure actually only require weaker structure, such as an $\mathrm{SO} \times G$-structure; this is believed to model the phenomenon in physics of phases which appear to be fermionic, but are in fact bosonic phases that are not fermionic in an interesting way. Finally, in §4.3.4, we gather some lemmas we use repeatedly in the coming sections. The reader interested in the computations can read §4.3.1 and §4.3.2, returning to the other sections later.

4.3.1. Some interesting phases to study. In §§4.4–4.5, we compute equivariant phase homology groups for many 2- and 3-dimensional point groups. Using Ansatz 4.1.19, these computations yield predictions of groups of invertible topological phases. This is a lot of data, so we take the opportunity here to highlight which of our predictions would be interesting to study by other means, e.g. by arguing on the lattice.

We first study some cases already present in the literature and find agreement, including reflections in Altland-Zirnbauer classes D and A (§4.4.1), inversions in classes D and A (§4.4.2), cyclic groups acting by rotations in classes D and A (§4.4.3), and dihedral groups acting by rotations and reflections in class D (§4.4.4). In all cases we consider both spinless and spin-1/2 fermions.

In addition, we study rotations in class A and many three-dimensional point group symmetries in classes D and A: dihedral groups acting by rotations, pyritohedral symmetry, and chiral and full tetrahedral, octahedral, and icosahedral symmetries. We consider symmetry types with both spinless and spin-1/2 fermions. To the best of our knowledge, these symmetry types have not been studied in the literature, so we indicate some of our predictions that might be interesting to study.

(1) In §4.4.4.3 and §4.4.4.4, we compute phase homology groups for the local systems of symmetry types corresponding to class A phases in which the dihedral group D_{2n} acts by rotations and reflections.
   (a) In dimension $d = 2$, we predict using Theorems 4.4.47 and 4.4.54 a phase generating a $\mathbb{Z}/2n$ for even n with spinless fermions.

(b) In dimension $d = 3$, we would be interested in the predicted $\mathbb{Z}/8 \oplus \mathbb{Z}/2$ for n odd, with either spin-1/2 or spinless fermions (based on (4.4.46), Theorem 4.4.46), as well as a phase generating a $\mathbb{Z}/4$ for n even with spin-1/2 fermions (based on Theorems 4.4.58 and 4.4.61).

(2) We predict using §4.5.2 a $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ of 3d class D phases with a pyritohedral symmetry and spinless fermions. In class A, we predict a phase generating a $\mathbb{Z}/4$ subgroup, again with spinless fermions.

(3) In §4.5.1, we calculate equivariant phase homology groups on $\mathbb{R}^3$ for A_4 acting by tetrahedral symmetry and find that for classes A and D and the spinless and spin-1/2 cases, the zeroth phase homology groups all vanish. Under our ansatz, this predicts there are no nontrivial fermionic phases equivariant for a tetrahedral symmetry in these cases. Can this be seen using a lattice argument?

(4) We predict in §4.5.3.1 that for 3d class D phases with a full tetrahedral symmetry (i.e. including reflections) and spinless fermions, there is a phase generating a $\mathbb{Z}/4$ subgroup. This phase homology calculation required the most involved mathematical argument, and it would be interesting to see a physical description. A physical interpretation of Proposition 4.5.44 specifically or an argument averting it would provide some insight into the meaning in physics of the Adams spectral sequence as a tool for studying fermionic phases.

Our computations predict plenty of other phases, but many of them either have Adams filtration zero (see §4.3.3) and therefore are not predicted to be intrinsically fermionic, or have more complicated symmetry types, such a full octahedral symmetry, that would be harder to study on the lattice.

Remark 4.3.1. In the computations we make in the next several sections, we generally report more bordism groups than we need to determine the phase homology groups corresponding to groups of invertible phases: to compute the group of n-dimensional invertible field theories with symmetry type $H \to \mathrm{O}$, we need the torsion subgroup of $\pi_n(MTH)$ and the free summand in $\pi_{n+1}(MTH)$. Bordism has other applications in geometry and physics, so we usually report all bordism groups $\pi_k(MTH)$ that follow from the calculations that we need for crystalline phases. When $k \geq n+1$, these provide information about higher-dimensional crystalline phases; for $k < \dim(\lambda)$, though, it is not clear what a crystalline phase could mean when there are not enough space dimensions for G to act by λ, and we do not give a physical meaning to these computations. See [**GOP+20**] for some discussion when space*time* is $\dim(\lambda)$-dimensional.

4.3.2. Methods of computation. In this section, we summarize the techniques we use to make these computations, and gather a few auxiliary lemmas we need along the way. Most of our computations can be reframed as computing certain twisted *ko*- and *ku*-homology groups of finite groups in low degrees; the reader interested in learning how to perform such computations is encouraged to refer to the monographs of Bruner-Greenlees [**BG03, BG10**] on connective *ko*- and *ku*-theory, as well as Beaudry-Campbell's article [**BC18**] on using the Adams spectral sequence to compute *ko*-theory.

Computing spin bordism: Let ko denote the connective real K-theory spectrum. Anderson-Brown-Peterson [**ABP67**] show that the Atiyah-Bott-Shapiro map $MTSpin \to ko$ [**ABS64**] is 7-connected, meaning that for any space or spectrum X, the induced map $\Omega_k^{\mathrm{Spin}}(X) \to ko_k(X)$ is an isomorphism for $k \le 7$. We often pass between spin bordism and *ko*-theory without comment. We compute the free and 2-torsion summands of $ko_*(X)$ using the Adams spectral sequence; see below. The forgetful map $MTSpin \to MTSO$ induces an equivalence on odd-primary torsion, so to compute odd-primary torsion, we typically compute $\Omega_*^{\mathrm{SO}}(X)$ via the Atiyah-Hirzebruch spectral sequence, which we also discuss below.

Computing spin^c bordism: Let ku denote connective complex K-theory. Anderson-Brown-Peterson (*ibid.*) also produce a 7-connected map $MTSpin^c \to ku \vee \Sigma^4 ku$; we will also use the Adams spectral sequence to determine the free and 2-torsion summands of $ku_*(X)$, as described below. The forgetful map $MTSpin^c \to MTSO \wedge (B\mathrm{U}_1)_+$ induces an equivalence on odd-primary torsion, so we compute $\Omega_*^{\mathrm{SO}}(X \times B\mathrm{U}_1)$, typically with the Atiyah-Hirzebruch spectral sequence.

4.3.3. Adams filtration 0 phases are secretly bosonic. In Remark 4.1.25, we defined a map from phase homology with symmetry type SO to phase homology with symmetry types Spin or Spin^c and interpreted it as regarding bosonic SPT phases as fermionic SPT phases in a trivial way. Physicists studying fermionic SPT phases are often interested in the cokernel of this map, which is thought of as the group of intrinsically fermionic SPT phases. Because bosonic crystalline phases are relatively well-understood, e.g. in the work of Hermele, Huang, Song, and their collaborators [**HSHH17**, **SHFH17**, **HH18**, **SHQ$^+$19**, **SFQ20**, **SXH20**] and via the bosonic crystalline equivalence principle of Thorngren-Else [**TE18**], we are most interested in intrinsically fermionic SPT phases.

The structure of the Adams spectral sequence allows us to identify the image of this bosonic-to-fermionic map on phase homology with little extra work. For more about the Adams spectral sequence, see §4.3.2; for now, we need only that phase homology groups, reinterpreted through Theorem 4.2.8 as groups of invertible field theories, are computed as homotopy groups of spectra, and that the homotopy groups of any spectrum M come with a canonical filtration called the *(mod 2) Adams filtration*

$$\pi_n M = F_n^0 \supseteq F_n^1 \supseteq F_n^2 \supseteq \cdots \tag{4.3.2}$$

For more information, see [**BC18**, §4.7]. This has two properties which are important for us.

(1) The Adams spectral sequence computes the Adams filtration: after 2-completing, the associated graded of (4.3.2) is the E_∞-page of the Adams spectral sequence, in that $E_\infty^{s,t} = \mathrm{gr}_s\pi_{t-s}M$.

(2) If $M = MTH$ is a Thom spectrum whose homotopy groups compute bordism groups, elements of the associated graded in degree 0 correspond to the 2-primary part of the group of deformation classes of invertible TFTs which depend on something weaker than an H-structure, such as a spin IFT which is defined by evaluating an oriented IFT on spin manifolds.

This means we can identify which invertible TFTs really use the H-structure, and which do not.

Now a little more detail. We do not need to say much more about (1): we depict Adams spectral sequences on a grid with coordinates $(t-s, s)$, such as in Figure 7, right, so F_n^0/F_n^1 is found in the E_∞-page at coordinate $(n, 0)$.

For (2), we make a simplifying assumption: that for the specific degree n we are investigating, $\pi_n MTH$ is 2-torsion. This assumption holds in all cases where we want to study the Adams filtration in this article, but if you want to relax it, see Remark 4.3.11. The assumption implies that up to extension questions on the E_∞-page, the mod 2 Adams spectral sequence fully determines $\pi_n MTH$,[11] and that the natural map

$$(\pi_n(MTH))^\vee := \mathrm{Hom}(\pi_n(MTH), \mathbb{C}^\times) \longrightarrow [MTH, \Sigma^{n+1}I_{\mathbb{Z}}] \tag{4.3.3}$$

is an isomorphism.

To pass from bordism groups to isomorphism class of invertible field theories, we must take character duals $A \mapsto A^\vee := \mathrm{Hom}(A, \mathbb{C}^\times)$. This is a good thing, actually: a degree-0 element of $\mathrm{gr}_\bullet\pi_n(MTH)$ does not usually uniquely lift to an element of $\pi_n MTH$: the ambiguity is F_n^1. But in $(\pi_n(MTH))^\vee$, we get a subgroup: the surjection

$$\pi_n(MTH) \twoheadrightarrow \pi_n(MTH)/F_n^1 \cong \mathrm{gr}_0\pi_n(MTH) \tag{4.3.4a}$$

passes under character duality to an inclusion

$$(\mathrm{gr}_0\pi_n(MTH))^\vee \hookrightarrow (\pi_n(MTH))^\vee. \tag{4.3.4b}$$

[11] Some extension questions can be addressed using the $H^{*,*}(\mathcal{A}(1))$-action on the E_∞-page, but there are also *hidden extensions* which are harder to address. None of the calculations we make in this article manifest hidden non-split extensions; one example where they do occur is $H = \mathrm{Spin} \times_{\mathbb{Z}/2} \mathbb{Z}/8$ [**DDHM**].

Therefore, in a mild abuse of notation, we refer to this subgroup of $(\pi_n(MTH))^\vee$, identified with a subgroup of the group isomorphism classes of invertible TFTs with H-structure, as the group of *Adams filtration* 0 *invertible TFTs with H-structure.*

It is a theorem [**FH19b**, §8.4] that this subgroup consists of theories closely related to classical Dijkgraaf-Witten theories [**FQ93**, §1].[12] Isomorphism classes of these invertible TFTs are determined by their partition functions [**FH16a**, §5.3], so we specify these theories by their partition functions, which are bordism invariants $\Omega_n^H \to \mathbb{C}^\times$.

For the Adams spectral sequence, $E_2^{0,n} = \mathrm{Ext}_{\mathcal{A}}^{0,n}(\widetilde{H}^*(MTH;\mathbb{Z}/2);\mathbb{Z}/2)$ is canonically identified with

$$\mathrm{Hom}_{\mathcal{A}(1)}(\widetilde{H}^*(MTH;\mathbb{Z}/2), \Sigma^n\mathbb{Z}/2), \tag{4.3.5}$$

which is a subspace of

$$\mathrm{Hom}_{\mathsf{Ab}}(\widetilde{H}^n(MTH;\mathbb{Z}/2), \mathbb{Z}/2) \cong (\widetilde{H}^n(MTH;\mathbb{Z}/2))^\vee. \tag{4.3.6}$$

The fourth quadrant of the Adams spectral sequence is empty, so $E_\infty^{0,n}$ is a subspace of $E_2^{0,n}$. Take the sequence of maps

$$\mathrm{gr}_0\pi_n(MTH) = E_\infty^{0,n} \hookrightarrow E_2^{0,n} \hookrightarrow (\widetilde{H}^n(MTH;\mathbb{Z}/2))^\vee \tag{4.3.7a}$$

and apply character duality:

$$(\mathrm{gr}_0\pi_n(MTH))^\vee \twoheadleftarrow (E_2^{0,n})^\vee \twoheadleftarrow \widetilde{H}^n(MTH;\mathbb{Z}/2). \tag{4.3.7b}$$

Now compose with the Thom isomorphism to obtain

$$\zeta\colon H^n(BH;\mathbb{Z}/2) \twoheadrightarrow (\mathrm{gr}_0\pi_n(MTH))^\vee. \tag{4.3.7c}$$

That is, a degree-n mod 2 cohomology class of BH determines an isomorphism class of Adams filtration 0 invertible TFTs, and all Adams filtration 0 invertible TFTs arise in this way. The map need not be injective, e.g. by the Wu formula when $H = \mathrm{O}$.

Tracing this through Thom's collapse map tells us that given a cohomology class $\theta \in H^n(BH;\mathbb{Z}/2)$, the partition function $\zeta(\theta)$ is the bordism invariant which takes a closed n-manifold with H-structure

[12]These theories are not quite the same thing as classical Dijkgraaf-Witten theories, which are TFTs of oriented manifolds with a principal G-bundle, and which use $\mathbb{R}/\mathbb{Z}$-valued cohomology, rather than $\mathbb{Z}/2$-valued cohomology. Unoriented generalizations of classical Dijkgraaf-Witten theory are studied in more detail in work of Kim [**Kim18**, §6] and You [**You20**], as well as Section 2.2 of this thesis.

$(M, f\colon M \to BH)$ and returns

$$\zeta(\theta)(M, f) = (-1)^{\langle f^*\theta, [M]\rangle}. \tag{4.3.8}$$

That is, use the H-structure to pull θ back to M, then evaluate it on the mod 2 fundamental class. This construction uses some aspects of the H-structure on M, but in the cases relevant to this paper, it is insensitive to the difference between Spin and O, which is believed to pass to the physicists' distinction between fermionic and bosonic phases.

Lemma 4.3.9. *If $H = \mathrm{Spin}\times_{\mu_2}\widetilde{G}$ or $H = \mathrm{Spin}^c\times_{\mu_2}\widetilde{G}$, where $\widetilde{G}$ is in Data 4.2.1, and $H' \coloneqq \mathrm{O}\times G$, then the map $H \to H'$ of tangential structures induces a surjective map $H^*(BH';\mathbb{Z}/2)\to H^*(BH;\mathbb{Z}/2)$, and therefore the partition functions (4.3.8) of the Adams filtration 0 theories only depend on the underlying H'-structure of an H-manifold.*

Proof. First, the Spin case. We established a shearing equivalence $MTH \cong MTSpin\wedge X$, where X is a Thom spectrum of a rank-zero virtual vector bundle over BG, and this equivalence fits into a homotopy commutative diagram

$$\begin{tikzcd} MTH \arrow[r, "\simeq"] \arrow[d] & MTSpin\wedge X \arrow[d] \\ MTO\wedge (BG)_+ \arrow[r] & MTO\wedge X. \end{tikzcd} \tag{4.3.10a}$$

Apply mod 2 cohomology and invoke the Thom isomorphism to obtain a commutative diagram

$$\begin{tikzcd} H^*(BH;\mathbb{Z}/2) & H^*(B\mathrm{Spin}\times BG;\mathbb{Z}/2) \arrow[l, "\cong"'] \\ H^*(B\mathrm{O}\times BG;\mathbb{Z}/2) \arrow[u] & H^*(B\mathrm{O}\times BG;\mathbb{Z}/2) \arrow[l, "\mathrm{id}"'] \arrow[u, "\xi"'] \end{tikzcd} \tag{4.3.10b}$$

The map $H^*(B\mathrm{O};\mathbb{Z}/2)\to H^*(B\mathrm{Spin};\mathbb{Z}/2)$ is surjective, so the Künneth formula implies ξ is too, so the left-hand arrow $H^*(BH';\mathbb{Z}/2)\to H^*(BH;\mathbb{Z}/2)$ is as well.

For $H = \mathrm{Spin}^c$, the proof is the same – $B\mathrm{Spin}^c$ has an additional characteristic class $c_1\in H^2(B\mathrm{Spin}^c;\mathbb{Z})$, but its mod 2 reduction is w_2, so ξ is still surjective. ☐

Remark 4.3.11. In all cases that one might reasonably encounter, the bordism group $\pi_n X$ is finitely generated, so we can ask what happens if it contains p-torsion for an odd prime p or free summands. For a p-torsion summand, the story is very similar: one instead uses the mod p Adams filtration on $\pi_n M_p^\wedge$, which is detected by the $\mathbb{Z}/p$-Adams spectral sequence. This has almost the same signature as the $\mathbb{Z}/2$-Adams spectral sequence we use in this paper, except that $\mathbb{Z}/2$ is replaced with $\mathbb{Z}/p$ and the Steenrod algebra is over $\mathbb{Z}/p$ instead of $\mathbb{Z}/2$. Because the mod p Thom isomorphism requires an orientation, the story is a little more nuanced for tangential structures which do not induce an orientation.

For free summands in $\pi_n M$, there is no analogous story. The invertible field theories in question are not topological, and at present their classification is still a conjecture [**Fre19**, Lecture 9]. Assuming this conjecture, though, the Adams filtration does not tell the whole story. For example, consider 3d invertible spin field theories, (conjecturally) classified by

$$[MTSpin, \Sigma^4 I_{\mathbb{Z}}] \xrightarrow{\cong} \mathrm{Hom}(\Omega_4^{\mathrm{Spin}}, \mathbb{Z}) \cong \mathbb{Z}, \tag{4.3.12}$$

generated by the map φ sending a spin 4-manifold to its signature divided by 16 [**Roh52**]. As the signature does not depend on the spin structure, 16φ generates $\mathrm{Hom}(\Omega_4^{\mathrm{SO}}, \mathbb{Z})$,[13] and therefore the image of the forgetful map $[MTSO, \Sigma^4 I_{\mathbb{Z}}] \to [MTSpin, \Sigma^4 I_{\mathbb{Z}}]$ is identified with the subgroup $16\mathbb{Z}$. That is, assuming the conjecture on the classification of not-necessarily-topological invertible field theories, a 3d spin invertible field theory only depends on the underlying orientation iff it is q times a generator, where $16 \mid q$. So for free summands in the abelian group of isomorphism classes of invertible field theories, the Adams filtration approach does not work, and one must use other methods.

4.3.4. A few utility lemmas. Recall from Definition 1.1.24 that we defined local systems A_α on a space X associated to an abelian group A and an element $\alpha \in H^1(X; \mathbb{Z}/2)$.

Proposition 4.3.13. *Let $\sigma \to B\mathbb{Z}/2$ denote the tautological line bundle.*

(1) $H^k(B\mathbb{Z}/2; \mathbb{Z}_{w_1(\sigma)})$ is isomorphic to $\mathbb{Z}/2$ in odd degrees and 0 in even degrees.

(2) If n is odd, $H^k(B\mathbb{Z}/2; (\mathbb{Z}/n)_{w_1(\sigma)}) \cong 0$ for all k.

(3) If n is even, $H^k(B\mathbb{Z}/2; (\mathbb{Z}/n)_{w_1(\sigma)}) \cong \mathbb{Z}/2$ for all k.

Proof. Use $\mathbb{RP}^\infty := \varinjlim_n \mathbb{RP}^n$ as our model for $B\mathbb{Z}/2$. Let A be any abelian group. Given k, choose a very large even m; then, the map $\mathbb{RP}^m \hookrightarrow B\mathbb{Z}/2$ induces an isomorphism $H^k(\mathbb{RP}^m; A_{w_1(\sigma)}) \xrightarrow{\cong} H^k(B\mathbb{Z}/2; A_{w_1(\sigma)})$. Since m is even, $\mathbb{RP}^m$ is unorientable, and $\mathbb{Z}_{w_1(\sigma)}$ is isomorphic to the orientation local system for $\mathbb{RP}^m$, so there is a Poincaré duality isomorphism $H^k(\mathbb{RP}^m; A_{w_1(\sigma)}) \cong H_{m-k}(\mathbb{RP}^m; A)$. $\square$

We will repeatedly use the following theorem to show some differentials and extensions are trivial in the Adams spectral sequence.

[13] This follows from the fact that the signature defines an isomorphism $\sigma\colon \Omega_4^{\mathrm{SO}} \to \mathbb{Z}$, which follows from the fact that $\mathbb{CP}^2$, with signature 1, generates Ω_4^{SO} [**Tho54**, Remarque following Corollaire IV.18].

Theorem 4.3.14 (Margolis [**Mar74**])**.** *Let $\mathcal{B}$ be a sub-Hopf algebra of Steenrod algebra and Y be a spectrum with $\widetilde{H}^*(Y;\mathbb{Z}/2) \cong \mathcal{A}\otimes_{\mathcal{B}}\mathbb{Z}/2$ (so that the change-of-rings trick works for computing 2-completed Y-homology). For any spectrum X, there is a splitting*

$$Y \wedge X \simeq F \vee \overline{X}, \tag{4.3.15}$$

where F is an Eilenberg-Mac Lane spectrum for a graded $\mathbb{Z}/2$-vector space and $\widetilde{H}^*(\overline{X};\mathbb{Z}/2)$ has no free summands as an $\mathcal{A}$-module.

The upshot is that in the Adams spectral sequence for computing $\pi_*(Y\wedge X)^\wedge_2$, the piece of the E_2-page coming from free summands of $\widetilde{H}^*(X;\mathbb{Z}/2)$ as a $\mathcal{B}$-module do not emit or receive nontrivial differentials, and do not participate in nontrivial extensions.

Lemma 4.3.16. *Let G be a finite group and $E \to BG$ be a rank-zero virtual vector bundle.*

1. *If $4 \mid n$, $\widetilde{ko}_n(BG^E)\otimes\mathbb{Q} \cong H_0(BG;\mathbb{Q}_{w_1(E)})$; if $4\nmid n$, $\widetilde{ko}_n(BG^E)$ is torsion.*
2. *The same is true for $\widetilde{ku}_n(BG^E)$, except divisibility by 4 is replaced by divisibility by 2.*

Proof. Atiyah-Hirzebruch [**AH61**] proved that the Chern character defines an equivalence

$$ch\colon ku\wedge H\mathbb{Q} \xrightarrow{\simeq} \bigvee_{k\geq 0}\Sigma^{2k}H\mathbb{Q}. \tag{4.3.17}$$

The Thom isomorphism theorem establishes that $\widetilde{H}_*(BG^E;\mathbb{Q}) \cong H_*(BG;\mathbb{Q}_{w_1(E)})$, and since G is finite, this vanishes above degree zero by Maschke's theorem.

The proof for ko-theory is the same, except first using the complexification map $c\colon ko\to ku$:

$$ch\circ c\colon ko\wedge H\mathbb{Q} \xrightarrow{\simeq} \bigvee_{k\geq 0}\Sigma^{4k}H\mathbb{Q}. \tag{4.3.18}$$

□

Choosing E to be the trivial bundle shows the conclusions also hold for the torsion in $\widetilde{ko}_*(BG)$ and $\widetilde{ku}_*(BG)$.

Lemma 4.3.19 (Adem-Milgram)**.** *Fix a prime p, and let H be a subgroup of a finite group G with $[G:H]$ coprime to p and P be a Sylow p-subgroup of H. Assume P is abelian and that $N_H(P)/P = N_G(P)/P$; then the restriction map $\rho_{H,G}\colon H^*(BG;\mathbb{Z}/p)\to H^*(BH;\mathbb{Z}/p)$ is an isomorphism.*

Proof. This is a slight strengthening of theorems of Swan [**Swa60**] and Adem-Milgram [**AM04**, Theorems II.6.6 and II.6.8], who prove that if K is a finite group with Abelian p-Sylow subgroup P, then the restriction map $H^*(BK;\mathbb{Z}/p) \to H^*(BP;\mathbb{Z}/p)^{N_K(P)}$ is an isomorphism. In our setting, the data of P and $N(P)/P$ are identical for G and H, so both restriction maps $r_{P,G}\colon H^*(BG;\mathbb{Z}/p) \to H^*(BP;\mathbb{Z}/p)^N$ and $r_{P,H}\colon H^*(BH;\mathbb{Z}/p) \to H^*(BP;\mathbb{Z}/p)^N$ are isomorphisms. Since $r_{P,G} = r_{P,H} \circ \rho_{G,H}$, we are done. □

Lemma 4.3.20 (Bock-to-Sq^1 lemma)**.** *Let $\beta\colon H^k(-;\mathbb{Z}/2) \to H^{k+1}(-;\mathbb{Z})$ denote the integral Bockstein. Then $\beta(x) \bmod 2 = \mathrm{Sq}^1(x)$.*

Proof. The commutative diagram of short exact sequences

(4.3.21)
$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "2"] & \mathbb{Z}/4 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

induces a commutative diagram of their induced long exact sequences in cohomology; in particular, $\beta \bmod 2$ equal to the Bockstein for the bottom short exact sequence, which is Sq^1. □

In the mixed unoriented case, Theorems 4.2.11 and 4.2.24 ask us to study Thom spectra for determinants of representations. We use the following lemma to simplify them.

Lemma 4.3.22. *Let $\lambda\colon G \to \mathrm{O}_d$ be a faithful representation whose image contains a reflection and $V_\lambda \to BG$ be the associated vector bundle. Then the splitting of the surjection*

(4.3.23)
$$G \xrightarrow{\lambda} \mathrm{O}_d \xrightarrow{\pi_0} \mathbb{Z}/2$$

lifts to a splitting of the Thom spectrum $(BG)^{\mathrm{Det}(V_\lambda)-1}$ as

(4.3.24)
$$(BG)^{\mathrm{Det}(V_\lambda)-1} \xrightarrow{\simeq} (B\mathbb{Z}/2)^{\sigma-1} \vee M,$$

and the inclusion $\widetilde{H}^*(M;\mathbb{Z}/2) \hookrightarrow \widetilde{H}^*((BG)^{\mathrm{Det}(V_\lambda)-1};\mathbb{Z}/2)$ is injective with image a complementary vector space to the subspace spanned by $\{Uw_1(V_\lambda)^k \mid k \geq 0\}$.

Proof. Let $g \in G$ be an element sent to λ by a reflection. Then $g^2 = 1$, so the maps $\langle g\rangle \hookrightarrow G \twoheadrightarrow \mathbb{Z}/2$ compose to an isomorphism. Upon taking Thom spectra, these can be identified with maps $(B\mathbb{Z}/2)^{\sigma-1} \to (BG)^{\mathrm{Det}(V_\lambda)-1} \to (B\mathbb{Z}/2)^{\sigma-1}$ composing to (a map homotopy equivalent to) the identity, which splits off $(B\mathbb{Z}/2)^{\sigma-1}$. The image of the map $\widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1};\mathbb{Z}/2) \to \widetilde{H}^*((BG)^{\mathrm{Det}(V_\lambda)-1};\mathbb{Z}/2)$ is spanned by $\{Uw_1(V_\lambda)^k \mid k \geq 0\}$, and the image of $\widetilde{H}^*(M;\mathbb{Z}/2)$ is a complementary subspace. □

4.4. Examples: rotations and reflections

4.4.1. Warmup: reflections. The simplest example of the fermionic crystalline equivalence principle occurs when the spatial symmetry is $\mathbb{Z}/2$ acting by a reflection. This symmetry can mix with $\mu_2 \subset \mathrm{Spin}_d$, and there are two cases. The following principle is well-established in physics literature; see Shiozaki-Shapourian-Ryu [**SSR17b**] and Song-Huang-Fu-Hermele [**SHFH17**, §VII].

- If $\mathbb{Z}/2$ and μ_2 do not mix (often written that the reflection squares to 1), then the classification matches the classification of pin^+ invertible field theories.
- Conversely, if $\mathbb{Z}/2$ and μ_2 do mix (often written that the reflection squares to $(-1)^F$), the classification matches that of pin^- invertible field theories.

Condensed-matter theorists also study theories with time-reversal symmetry. Though this is also an antiunitary symmetry that can mix with μ_2, the classification in terms of pin structures is opposite that of reflections: when time-reversal symmetry does not mix with fermion parity, we get pin^-, and when it does mix, we get pin^+. This is also well-established in physics, and is discussed by Kapustin-Thorngren-Turzillo-Wang [**KTTW15**], Freed-Hopkins [**FH16a**], and others.

The difference between these two correspondences is a first hint that the fermionic crystalline equivalence principle must be more complicated than the bosonic version; this point is raised by Thorngren-Else [**TE18**, §V.A] and Cheng-Wang [**CW18**, §II.C].

d	Class D, spinless §4.4.1.1	Class D, spin-1/2 §4.4.1.2	Class A §4.4.1.3
1	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$\mathbb{Z}/4$
2	$\mathbb{Z}/2$	0	0
3	$\mathbb{Z}/16$	0	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$
4	0	0	0

Table 1. $\mathbb{Z}/2$-equivariant phase homology groups for the cases in which $\mathbb{Z}/2$ acts by a reflection. As discussed in §4.4.1, these arise as the homotopy groups of the Anderson duals of $MTPin^+$, $MTPin^-$, and $MTPin^c$. For this group action, the spinless and spin-1/2 classifications in class A coincide.

4.4.1.1. *Class D, spinless.* When the reflection does not mix with the internal symmetry group, our ansatz is exactly that of Freed-Hopkins. In this setting, $\mathbb{Z}/2$ acts on $\mathbb{R}^d$ as $(d-1)+\sigma$, where k denotes the rank-k trivial representation and σ denotes the sign representation. Let f_0^D denote the equivariant local system of symmetry types for the class D spinless case. Arguing as in [**FH19a**, (3.6)], in space dimension d we see that

$$Ph_0^{\mathbb{Z}/2}(\mathbb{R}^d; f_0^D) \cong [MTSpin \wedge (B\mathbb{Z}/2)^{1-\sigma}, \Sigma^{d+2} I_{\mathbb{Z}}]. \tag{4.4.1}$$

Using (4.2.10a), $MTSpin \wedge (B\mathbb{Z}/2)^{1-\sigma} \simeq MTPin^+$, identifying these phase homology groups as homotopy groups of the Anderson dual of $MTPin^+$, as expected. Finally, to obtain the specific groups in Table 1, we use the preexisting calculations of pin$^+$ bordism from [**Gia73b, KT90a, KT90b**].

4.4.1.2. *Class D, spin-1/2.* Again $\mathbb{Z}/2$ acts by $d - 1 + \sigma$, and this time, reflection mixes with fermion parity. Let $f_{1/2}^D$ denote the equivariant local system of symmetry types for this case. The associated bundle to the $\mathbb{Z}/2$-representation given by reflection is not pin$^-$, so by Theorem 4.2.11,

$$Ph_0^{\mathbb{Z}/2}(\mathbb{R}^d; f_{1/2}^D) \cong [MTSpin \wedge (B\mathbb{Z}/2)^{\mathrm{Det}(\sigma)-1}, \Sigma^{d+2} I_{\mathbb{Z}}]. \tag{4.4.2}$$

Because σ is a line bundle, $\mathrm{Det}(\sigma) = \sigma$. Using (4.2.10b), $MTSpin \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq MTPin^-$, so these phase homology groups are identified with homotopy groups of the Anderson dual of $MTPin^-$ as predicted. These bordism groups are calculated in [**ABP69, KT90b**].

4.4.1.3. *Class A.* For spin^c phases (those of Altland-Zirnbauer class A), the spinless and spin-1/2 classifications coincide: V_λ is pin^c, so Theorem 4.2.24 tells us to consider $MTSpin^c \wedge (B\mathbb{Z}/2)^{1-\sigma}$ in both cases, and by (4.2.10c), this spectrum is equivalent to $MTPin^c$.

Bahri-Gilkey [**BG87a, BG87b**] compute pin^c bordism groups,[14] giving us the phase homology groups in Table 1.

4.4.1.4. *Comparison with prior work.* Reflection-equivariant fermionic phases have been studied by many teams of researchers with many methods. Their results agree with each other, and with us.

Class D, spinless: These phases, especially the $\mathbb{Z}/16$ in $d = 3$, are studied by Song-Huang-Fu-Hermele [**SHFH17**, §V.A], Hsieh-Cho-Ryu [**HCR16**, §IV], Shiozaki-Shapourian-Ryu [**SSR17b**, §II.B, §II.D], Guo-Ohmori-Putrov-Wan-Wang [**GOP+20**, §10.7], and Mao-Wang [**MW20**].

Class D, spin-1/2: These phases have been studied by Song-Huang-Fu-Hermele [**SHFH17**, §V.B], Shapourian-Shiozaki-Ryu [**SSR17a, SSR17b**], Guo-Ohmori-Putrov-Wan-Wang [**GOP+20**, §10.7], and Bultinck-Williamson-Haegeman-Verstraete [**BWHV17**, §IX].

Class A: These phases have been studied by Isobe-Fu [**IF15**], Hong-Fu [**HF17**], Shapourian-Shiozaki-Ryu [**SSR17a, SSR17b**], Song-Huang-Fu-Hermele [**SHFH17**, §4], and Shiozaki-Shapourian-Gomi-Ryu [**SSGR18**, §V].

[14] In low degrees, Beaudry-Campbell [**BC18**, §5.6] compute low-degree pin^c bordism groups using the Adams spectral sequence over $\mathcal{A}(1)$, using that $MTPin^c \simeq MTSpin \wedge \Sigma^{-2} MU_1 \wedge \Sigma^{-1} MO_1$. One can also compute using the Adams spectral sequence over $\mathcal{E}(1)$, as in §4.4.4.3; we found this to be a fun and useful exercise for getting comfortable with this variation of the Adams spectral sequence.

4.4.2. Inversions. *Inversion symmetry* is the $\mathbb{Z}/2$-symmetry on $\mathbb{R}^d$ acting by

$$(x_1, \ldots, x_d) \longmapsto (-x_1, \ldots, -x_d). \tag{4.4.3}$$

This offers another relatively simple example of the FCEP, but with a new feature in the spin-1/2 case: the classes in $H^2(B\mathbb{Z}/2; \mathbb{Z}/2)$ specified by the extension $1 \to \mathbb{Z}/2 \to \widetilde{G} \to \mathbb{Z}/2 \to 1$ and by $w_2(\lambda) + w_1(\lambda)^2$ are not always equal. This does not change very much, as we explain in §4.4.2.2 below.

d	Class D, spinless §4.4.2.1	Class D, spin-1/2 §4.4.2.2	Class A §4.4.2.3
1	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$\mathbb{Z}/4$
2	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}/8$	$\mathbb{Z}^2 \oplus \mathbb{Z}/4$
3	0	$\mathbb{Z}/16$	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$
4	0	$\mathbb{Z} \oplus \mathbb{Z}/16$	$\mathbb{Z}^2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$

TABLE 2. $\mathbb{Z}/2$-equivariant phase homology groups for the cases where $\mathbb{Z}/2$ acts as inversion. The symmetry type whose Thom spectrum determines these groups depends on d; see the referenced sections for which symmetry types appear.

4.4.2.1. *Class D, spinless case.* First, the case for which inversion symmetry and fermion parity do not mix. The $\mathbb{Z}/2$-action on $\mathbb{R}^d$ is a direct sum of d copies of the sign representation σ, so as a $\mathbb{Z}/2$-space, $\mathbb{R}^d$ is denoted $d\sigma$. This case is covered by Freed-Hopkins [**FH19a**, Example 3.5], and the phase homology groups are

$$[\mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{d-d\sigma}, \Sigma^{d+2} I_{\mathbb{Z}}]. \tag{4.4.4}$$

The spectra $\mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{d-d\sigma}$ are periodic in d.

Lemma 4.4.5. *If $d' - d$ is divisible by 4, $\mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{d(1-\sigma)} \simeq \mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{d'(1-\sigma)}$.*

PROOF. This is an instance of Theorem 4.1.36, using that spin structures satisfy the 2-out-of-3 property and that, since 4σ is spin, so is $(d' - d)(1 - \sigma)$. □

Thus we have only to determine $\mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{d(1-\sigma)}$ for small d.

- When $d = 0$, we get $\mathit{MTSpin} \wedge (B\mathbb{Z}/2)_+$.
- When $d = 1$, (4.2.10a) tells us $\mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{1-\sigma} \simeq \mathit{MTPin}^+$.
- For $d = 2$, we have $\mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{2-2\sigma}$.[15]

[15] Campbell [**Cam17**, §7.8] shows this spectrum is equivalent to $MT(\mathrm{Spin} \times_{\mathbb{Z}/2} \mathbb{Z}/4)$. Bordism for this symmetry type, called spin-$\mathbb{Z}/4$ bordism or spin$^{c/2}$ bordism, is used in several places in recent mathematical physics literature, including [**Cam17**, **Hsi18**, **FH19a**, **GEM19**, **TY19**, **DL20a**, **GOP$^+$20**, **HKT20**, **WW20a**, **Wan20**, **MV21**].

- When $d = -1$, (4.2.10b) gives $\mathit{MTSpin} \wedge \Sigma^{-1}(B\mathbb{Z}/2)^{\sigma-1} \simeq \mathit{MTPin}^-$.

The low-degree homotopy groups of these spectra that we need are computed by Giambalvo [**Gia73b**] and Kirby-Taylor [**KT90a, KT90b**] (the pin$^+$ case); Anderson-Brown-Peterson [**ABP69**] and Kirby-Taylor [**KT90b**] (the pin$^-$ case); Giambalvo [**Gia73a**] (the case $d = 2$); and Mahowald-Milgram [**MM76**] (the spin $\times\, \mathbb{Z}/2$ case). Thus we obtain the phase homology groups for the spinless class D case in Table 2.

4.4.2.2. *Class D, spin-*$1/2$ *case.* Now we consider the case where the inversion symmetry and $\mu_2 \subset \mathrm{Pin}_d^-$ mix as specified by the nontrivial extension $1 \to \mu_2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 1$. This is not classified by $w_2 + w_1^2$ of the associated bundle to the spatial representation: in the language of §4.2, $\lambda \ncong \xi$. Instead, this extension is classified by $w_2(\sigma) + w_1(\sigma)^2$, and σ is not pin$^-$, so if $f_{1/2}^D$ denotes the class D spin-1/2 equivariant local system of symmetry types on $\mathbb{R}^d$, Theorem 4.2.11 computes $Ph_*^{\mathbb{Z}/2}(\mathbb{R}^d; f_{1/2}^D)$ using the Thom spectrum of the virtual bundle

$$-V \boxplus (\sigma + \sigma - d\sigma) \cong -V \boxplus (d-2)(1-\sigma). \tag{4.4.6}$$

Thus

$$Ph_0^{\mathbb{Z}/2}(\mathbb{R}^d; f_{1/2}^D) \cong [\mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{(d-2)(1-\sigma)}, \Sigma^{d+2} I_{\mathbb{Z}}], \tag{4.4.7}$$

and Lemma 4.4.5 says the domain is again 4-periodic, but differently from the spinless case.

- When $d = 0$, we have $\mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{2-2\sigma}$.
- When $d = 1$, we have $\mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq \mathit{MTPin}^-$.
- When $d = 2$, we have $\mathit{MTSpin} \wedge (B\mathbb{Z}/2)_+$.
- When $d = -1$, we have $\mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{1-\sigma} \simeq \mathit{MTPin}^+$.

In the degrees we need, these bordism groups are computed in the same references we gave above in §4.4.2.1, and the relevant phase homology groups appear in Table 2.

Remark 4.4.8. This fourfold periodicity in the tangential structure appears in a few other contexts in mathematical physics, such as recent work of Hason, Komargodski, and Thorngren [**HKT20**, §4.4] and Córdova, Ohmori, Shao, and Yan [**COSY20**] applying it to the study of anomalies of domain wall theories as well as work of Tachikawa and Yonekura [**TY19**, §3] studying anomalies arising in string theory.

4.4.2.3. *Class A.* In class A, whether with spinless or spin-1/2 fermions, the FCEP predicts by way of Theorem 4.2.24 that an inversion symmetry in dimension d leads us to study $\mathit{MTSpin}^c \wedge (B\mathbb{Z}/2)^{d-d\sigma}$. For any vector bundle $V \to X$, $V \oplus V \cong V \otimes \underline{\mathbb{C}}$, and complex vector bundles are spin^c, so by Theorem 4.1.36, we can remove factors of $2-2\sigma$ from $d-d\sigma$ without changing the Thom spectrum, so we want to study $\mathit{MTSpin}^c\wedge(B\mathbb{Z}/2)_+$ when d is even and $\mathit{MTSpin}^c\wedge(B\mathbb{Z}/2)^{1-\sigma}\simeq\mathit{MTPin}^c$ when d is odd.

We discussed pin^c bordism in §4.4.1.3. Bahri-Gilkey [**BG87a, BG87b**] also compute $\Omega_*^{\mathrm{Spin}^c}(B\mathbb{Z}/2)$: they establish that the *Smith homomorphism* $\widetilde\Omega_n^{\mathrm{Spin}^c}(B\mathbb{Z}/2)\to\Omega_{n-1}^{\mathrm{Pin}^c}$, which sends a spin^c manifold M and principal $\mathbb{Z}/2$-bundle $P\to M$ to the induced pin^c structure on a smooth submanifold representative of the Poincaré dual of $w_1(P)\in H^1(M;\mathbb{Z}/2)$, is an isomorphism for all n; thus we get the groups in Table 2 by applying the universal property (1.1.50) of $I_\mathbb{Z}$ to either $\Omega_*^{\mathrm{Pin}^c}$ or $\Omega_*^{\mathrm{Spin}^c}\oplus\Omega_{*-1}^{\mathrm{Pin}^c}$, depending on dimension.

4.4.2.4. *Comparison with prior work.* Inversion-symmetric SPT phases are pretty well-studied, even in the fermionic case, and our phase homology calculations reproduce classifications of inversion-symmetric phases in the literature.

Class D, spinless: These phases are studied by Shiozaki-Xiong-Gomi [**SXG18**, §V.B] as well as Cheng-Wang [**CW18**, §III].

Class D, spin-1/2: These phases are studied by You-Xu [**YX14**, §III], Shiozaki-Shapourian-Ryu [**SSR17a, SSR17b**], Cheng-Wang [**CW18**, §III], and Shiozaki-Xiong-Gomi [**SXG18**, §V.A].

Class A: These phases are studied by You-Xu [**YX14**, §IV.A.3], Shiozaki-Shapourian-Ryu [**SSR17b**, §V.B], and Song-Huang-Fu-Hermele [**SHFH17**, §IV]. Shiozaki-Shapourian-Ryu also study the phases corresponding to the $\mathbb{Z}/2^{k+2}$ summand in $[\mathit{MTPin}^c,\Sigma^{2k+3}I_\mathbb{Z}]$ in arbitrary odd dimensions.[16]

Remark 4.4.9. Guo-Ohmori-Putrov-Wan-Wang [**GOP$^+$20**, §10.8] also study inversion-symmetric fermionic phases from a bordism-theoretic perspective, in both the spinless and spin-1/2 cases. Their results disagree with ours, and with the rest of the literature, because they use different symmetry types to model inversion-equivariant fermionic phases.

4.4.3. Rotations. We turn to the case of phases equivariant for the cyclic group C_n acting by rotation on a plane. These phases have been studied by several groups of authors, and our results are consistent with prior work; see §4.4.3.4 for more information.

Let $\lambda\colon C_n\to\mathrm{SO}_2$ denote this representation and $V_\lambda\to BC_n$ be the associated vector bundle. One can directly check that $C_n\to\mathrm{SO}_2$ lifts across $\mathrm{Spin}_2\to\mathrm{SO}_2$ iff n is odd.

4.4.3.1. *Class D, spinless case.* In this case, C_n does not mix with $\mu_2\subset\mathrm{Spin}$, and Theorem 4.2.11 reduces Ansatz 4.1.19 to the computation of $[\mathit{MTSpin}\wedge(BC_n)^{2-V_\lambda},\Sigma^{d+2}I_\mathbb{Z}]$ if n is even, or $[\mathit{MTSpin}\wedge(BC_n)_+]$, if n is odd.

[16]The presence of this summand follows from the existence of a $\mathbb{Z}/2^{k+2}$ summand in $\Omega_{2k+2}^{\mathrm{Pin}^c}$, which is proven by Bahri-Gilkey [**BG87b**].

d	n	Class D, spinless §4.4.3.1	Class D, spin-1/2 §4.4.3.2	Class A §4.4.3.3
2	0 mod 4	$\mathbb{Z}\oplus\mathbb{Z}/(n/2)$	$\mathbb{Z}\oplus\mathbb{Z}/2n\oplus\mathbb{Z}/2$	$\mathbb{Z}^2\oplus\mathbb{Z}/2n\oplus\mathbb{Z}/(n/2)$
	2 mod 4	$\mathbb{Z}\oplus\mathbb{Z}/(n/2)$	$\mathbb{Z}\oplus\mathbb{Z}/4n$	$\mathbb{Z}^2\oplus\mathbb{Z}/2n\oplus\mathbb{Z}/(n/2)$
	$1,3$ mod 4	$\mathbb{Z}\oplus\mathbb{Z}/n$	$\mathbb{Z}\oplus\mathbb{Z}/n$	$\mathbb{Z}^2\oplus\mathbb{Z}/n\oplus\mathbb{Z}/n$
3	0 mod 4	0	0	0
	2 mod 4	0	0	0
	$1,3$ mod 4	0	0	0

TABLE 3. C_n-equivariant phase homology groups for the cases in which C_n acts by rotations. Classification of fermionic phases with a C_n rotation symmetry. For the spinless class D case, these are classified by $[\mathit{MTSpin}\wedge(BC_n)^{2-V_\lambda},\Sigma^{d+1}I_{\mathbb{Z}}]$; for spin-1/2 class D, by $[\mathit{MTSpin}\wedge(BC_n)_+,\Sigma^{d+1}I_{\mathbb{Z}}]$; and for class A, both spinless and spin-1/2, by $[\mathit{MTSpin}^c\wedge(BC_n)_+,\Sigma^{d+1}I_{\mathbb{Z}}]$.

Lemma 4.4.10. *$\Omega_3^{\mathrm{SO}}(BC_n)\cong\mathbb{Z}/n$, $\Omega_4^{\mathrm{SO}}(BC_n)\cong\mathbb{Z}$, and $\Omega_5^{\mathrm{SO}}(BC_n)$ is torsion.*

PROOF. Compute with the Atiyah-Hirzebruch spectral sequence for oriented bordism; it collapses for $p+q\le 4$, and the 5-line of the E^2-page is torsion, implying $\Omega_5^{\mathrm{SO}}(BC_n)$ is torsion. ☐

Corollary 4.4.11 (Bruner-Greenlees [**BG10**, Example 7.3.2, §12.2.D], García-Etxebarria-Montero [**GEM19**, §C.2])**.** *For n odd, $\Omega_3^{\mathrm{Spin}}(BC_n)\cong\mathbb{Z}/n$, $\Omega_4^{\mathrm{Spin}}(BC_n)\cong\mathbb{Z}$, and $\Omega_5^{\mathrm{Spin}}(BC_n)$ is torsion.*

PROOF. Because n is odd, BC_n is stably trivial at 2, and $\mathit{MTSpin}\to\mathit{MTSO}$ is an equivalence away from 2. ☐

Theorem 4.4.12. *If n is even, $\widetilde{\Omega}_3^{\mathrm{Spin}}((BC_n)^{2-V_\lambda})\cong\mathbb{Z}/(n/2)$ and $\widetilde{\Omega}_4^{\mathrm{Spin}}((BC_n))^{2-V_\lambda})\cong\mathbb{Z}$. The group $\widetilde{\Omega}_5^{\mathrm{Spin}}((BC_n)^{2-V_\lambda})$ is torsion.*

PROOF. The computation breaks into 2-primary and odd-primary pieces. The forgetful map $\Omega_*^{\mathrm{Spin}}\to\Omega_*^{\mathrm{SO}}$ is an odd-primary isomorphism, and because $2-V_\lambda$ is orientable, there is a Thom isomorphism $\widetilde{\Omega}_*^{\mathrm{SO}}((BC_n)^{2-V_\lambda})\cong\Omega_*^{\mathrm{SO}}(BC_n)$. Thus, Lemma 4.4.10 takes care of the odd-primary part.

Write $n=2^\ell m$, where m is odd. Then the map $BC_{2^\ell}\to BC_n$ is a stable 2-primary equivalence, because it induces an isomorphism on mod 2 cohomology, so for the 2-primary piece it suffices to understand the case $n=2^\ell$. Campbell [**Cam17**, Theorem 1.8] studies $\Omega_d^{\mathrm{Spin}}((BC_{2^\ell})^{2-V_\lambda})$, obtaining $\mathbb{Z}/2^{\ell-1}$ when $d=3$, $\mathbb{Z}$ when $d=4$, and torsion when $d=5$, which suffices.[17] ☐

[17] There are a few other computations of $\widetilde{\Omega}_*^{\mathrm{Spin}}((BC_{2^\ell})^{2-V_\lambda})$ in low degrees by other methods. For $\ell=1$, see Giambalvo [**Gia73b**], García-Etxebarria and Montero [**GEM19**, (C.21)], and Freed-Hopkins [**FH19a**, §5]. For $\ell>1$, see Botvinnik-Gilkey [**BG97**, §5] and Davighi-Lohitsiri [**DL20a**, §A.4]; Botvinnik-Gilkey only report the orders of the bordism groups, but their computations show that the groups we need are cyclic. Be aware that Campbell and Davighi-Lohitsiri consider a different vector bundle than $2-V_\lambda$, though their calculations apply to this case.

4.4.3.2. *Class D, spin-1/2 case.* Theorem 4.2.11 asks us to compute $[\mathit{MTSpin} \wedge (BC_n)_+, \Sigma^{d+2} I_{\mathbb{Z}}]$, which (1.1.50) tells us in terms of $\Omega_*^{\mathrm{Spin}}(BC_n)$. For n odd, we already saw this in Corollary 4.4.11.

Proposition 4.4.13. *Let $n \equiv 2 \bmod 4$. Then $\Omega_3^{\mathrm{Spin}}(BC_n) \cong \mathbb{Z}/4n$, $\Omega_4^{\mathrm{Spin}}(BC_n) \cong \mathbb{Z}$, and $\Omega_5^{\mathrm{Spin}}(BC_n)$ is torsion.*

Proof. Inclusion $BC_2 \to BC_n$ is a 2-local equivalence, so the fact that the 2-torsion is $\mathbb{Z}/8$ in degree 3 and vanishes in degree 4 follows as soon as we know that for $\Omega_*^{\mathrm{Spin}}(BC_2)$. This was originally done by Mahowald-Milgram [**MM76**] but has been computed in a few other places, including Mahowald [**Mah82**, Lemma 7.3], Bruner-Greenlees [**BG10**, Example 7.3.1], Siegemeyer [**Sie13**, Theorem 2.1.5], and García-Etxebarria and Montero [**GEM19**, (C.18)]. What remains is odd-primary information, which is equivalent to the odd-primary part of oriented bordism, which we computed in Lemma 4.4.10. □

Proposition 4.4.14. *For $n \equiv 0 \bmod 4$, $\Omega_3^{\mathrm{Spin}}(BC_n) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2n$, $\Omega_4^{\mathrm{Spin}}(BC_n) \cong \mathbb{Z}$, and $\Omega_5^{\mathrm{Spin}}(BC_n)$ is torsion.*

Proof. Write $n = 2^\ell m$, where m is odd. As in the proof of Theorem 4.4.12, the 2-primary part of the answer is detected by $BC_{2^\ell} \to BC_n$, and the odd-primary part of the answer is detected oriented bordism. Davighi-Lohitsiri [**DL20a**, §A.3] compute $\Omega_k^{\mathrm{Spin}}(BC_{2^\ell})$ for $k \le 6$, giving the 2-primary summand, and for the odd-primary part we use Lemma 4.4.10. □

Botvinnik-Gilkey-Stolz [**BGS97**, Theorem 2.4], Bruner-Greenlees [**BG10**, Example 7.3.3], and Siegemeyer [**Sie13**, §2.2] do special cases of this computation, by a variety of methods.

4.4.3.3. *Class A.* The representation of C_n on $\mathbb{R}^2$ by rotations is unitary (under the standard identification $\mathbb{R}^2 = \mathbb{C}$), hence spin^c, so in both the spinless and spin-1/2 cases, we consider $\mathit{MTSpin}^c \wedge (BC_n)_+$: in the spinless case, we have a Thom isomorphism $\mathit{MTSpin}^c \wedge (BC_n)^{2-V_\lambda} \xrightarrow{\simeq} \mathit{MTSpin}^c \wedge (BC_n)_+$, and in the spin-1/2 case, $\mathrm{Det}(V_\lambda)$ is trivial, so Theorem 4.2.24 also gives us $\mathit{MTSpin}^c \wedge (BC_n)_+$.

Theorem 4.4.15. *The first few spin^c bordism groups of BC_n are*

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}^c}(BC_{2k}) &\cong \mathbb{Z} & \Omega_0^{\mathrm{Spin}^c}(BC_{2k+1}) &\cong \mathbb{Z}\\
\Omega_1^{\mathrm{Spin}^c}(BC_{2k}) &\cong \mathbb{Z}/2k & \Omega_1^{\mathrm{Spin}^c}(BC_{2k+1}) &\cong \mathbb{Z}/(2k+1)\\
\Omega_2^{\mathrm{Spin}^c}(BC_{2k}) &\cong \mathbb{Z} & \Omega_2^{\mathrm{Spin}^c}(BC_{2k+1}) &\cong \mathbb{Z}\\
\Omega_3^{\mathrm{Spin}^c}(BC_{2k}) &\cong \mathbb{Z}/4k \oplus \mathbb{Z}/k & \Omega_3^{\mathrm{Spin}^c}(BC_{2k+1}) &\cong (\mathbb{Z}/(2k+1))^{\oplus 2}\\
\Omega_4^{\mathrm{Spin}^c}(BC_{2k}) &\cong \mathbb{Z}^2 & \Omega_4^{\mathrm{Spin}^c}(BC_{2k+1}) &\cong \mathbb{Z}^2,
\end{aligned}$$

and $\Omega_5^{\mathrm{Spin}^c}(BC_n)$ *is torsion for all* n.

Proof. Write $n = 2^\ell \cdot m$, where m is odd. It suffices to compute the 2-primary piece and $\Omega_*^{\mathrm{Spin}^c}(BC_n) \otimes \mathbb{Z}[1/2]$. The inclusion $C_{2^\ell} \to C_n$ is stably a 2-primary equivalence, so for the 2-primary piece it suffices to determine $\Omega_*^{\mathrm{Spin}^c}(BC_{2^\ell})$. Bahri-Gilkey [**BG87b**, Theorem 1] compute these groups; when $\ell = 0$ they are $\Omega_*^{\mathrm{Spin}^c}(\mathrm{pt})$, which begins $\mathbb{Z}$, 0, $\mathbb{Z}$, 0, $\mathbb{Z}^2$, 0; and when $\ell \neq 0$ we have the same free summands as when $\ell = 0$, but additional torsion summands: $\Omega_1^{\mathrm{Spin}^c}(BC_{2^\ell}) \cong \mathbb{Z}/2^\ell$, and $\Omega_3^{\mathrm{Spin}^c}(BC_{2^\ell}) \cong \mathbb{Z}/2^{\ell-1} \oplus \mathbb{Z}/2^{\ell+1}$.

After smashing with $H\mathbb{Z}[1/2]$, the forgetful map $\mathit{MTSpin}^c \to \mathit{MTSO} \wedge (B\mathrm{U}_1)_+$ is an equivalence, so $\mathit{MTSO} \wedge (B\mathrm{U}_1)_+$ detects all odd-primary torsion in spin^c bordism. To compute this, we use the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(B\mathrm{U}_1 \times BC_n; \Omega_q^{\mathrm{SO}}(\mathrm{pt})) \Longrightarrow \Omega_{p+q}^{\mathrm{SO}}(B\mathrm{U}_1 \times BC_n). \tag{4.4.16}$$

The Künneth theorem implies the first few homology groups of $B\mathrm{U}_1 \times BC_n$ are $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}/n$, $H_2 = \mathbb{Z}$, $H_3 = (\mathbb{Z}/n)^{\oplus 2}$, $H_4 = \mathbb{Z}$, and $H_5 = (\mathbb{Z}/n)^{\oplus 3}$. When we feed this to the spectral sequence (4.4.16), there are no nonzero differentials to or from any element in total degree $p + q < 5$: because $\Omega_i^{\mathrm{SO}} = 0$ for $i = 1, 2, 3$, the only possible nonzero differential would be a $d_4\colon E^2_{5,0} \to E^2_{0,4}$, but the splitting $\Omega_*^{\mathrm{SO}}(B\mathrm{U}_1 \times BC_n) = \Omega_*^{\mathrm{SO}}(\mathrm{pt}) \oplus \widetilde{\Omega}_*^{\mathrm{SO}}(B\mathrm{U}_1 \times BC_n)$ splits off the $q = 0$ line splits off from the rest of the spectral sequence, killing this d_4. This tells the odd-primary torsion in degrees 0 through 4, and since the 5-line of the E_2-page is torsion, $\Omega_5^{\mathrm{SO}}(B\mathrm{U}_1 \times BC_n)$ is also torsion. ∎

4.4.3.4. *Comparison with prior work.* Rotation-equivariant phases in class D have been studied by several groups, including Shiozaki-Shapourian-Ryu [**SSR17b**, §IV.C], Guo-Ohmori-Putrov-Wan-Wang [**GOP$^+$20**, §10.9], and Freed-Hopkins [**FH19a**, §5], who all restrict to the case $n = 2$, and most comprehensively by Cheng-Wang [**CW18**, §IV, §V], who consider arbitrary n and both the spinless and spin-1/2 cases in $d = 2, 3$. Freed-Hopkins begin from the same ansatz as us so agreement is no surprise. In the remaining cases, there is almost complete agreement: all classifications compute the same torsion summands, but they all miss the free summand in $d = 2$. This is not a discrepancy, however: many authors restrict to considering phases whose low-energy effective theories are expected to be topological field theories, which in the ansatz of Freed-Hopkins [**FH16a**, §§5.3–5.4] amounts to considering the torsion subgroup of the classification using $I_{\mathbb{Z}}\mathit{MTH}$. The non-topological theories corresponding to the free summand have been discussed in a few references, including Freed [**Fre19**, Lecture 9] and Wan-Wang [**WW20a**, §7.1]; at present, their mathematical description remains partly conjectural.

Rotation-equivariant phases in class A are studied by Shiozaki-Shapourian-Ryu [**SSR17b**, §IV.D], Shiozaki-Xiong-Gomi [**SXG18**, §V.C.1], and Lu-Vishwanath-Khalaf [**LVK19**]. Shiozaki-Shapourian-Ryu and Lu-Vishwanath-Khalaf's classifications agree with us on torsion but miss the free summand as before, and Shiozaki-Xiong-Gomi's computation completely matches ours. Again, the free summand corresponds to non-topological invertible field theories.

4.4.4. Rotations and reflections. In this section, we compute the phase homology groups corresponding to phases on $\mathbb{R}^d$ equivariant for the D_{2n}-action of rotations and reflections in a given plane. Zhang-Wang-Yang-Qi-Gu [**ZWY⁺20**] also study these phases for $d = 2$ and in class D; we compare our results to theirs in §4.4.4.5.

d	n	Class D, spinless §4.4.4.1	Class D, spin-1/2 §4.4.4.2	Class A, spinless §4.4.4.3	Class A, spin-1/2 §4.4.4.4
2	0 mod 4	$(\mathbb{Z}/2)^{\oplus 2}$	$(\mathbb{Z}/2)^{\oplus 2}$	$\mathbb{Z}/2n$	$\mathbb{Z}/(n/2) \oplus (\mathbb{Z}/2)^{\oplus 2}$
	2 mod 4	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{\oplus 2}$	$\mathbb{Z}/2n$	$\mathbb{Z}/n \oplus \mathbb{Z}/2$
	$1, 3$ mod 4	$\mathbb{Z}/2$	0	$\mathbb{Z}/n$	$\mathbb{Z}/n$
3	0 mod 4	$(\mathbb{Z}/2)^{\oplus 4}$	0	$(\mathbb{Z}/2)^{\oplus 4}$	$\mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$
	2 mod 4	$(\mathbb{Z}/2)^{\oplus 3}$	0	$(\mathbb{Z}/2)^{\oplus 4}$	$\mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$
	$1, 3$ mod 4	$\mathbb{Z}/16$	0	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$

TABLE 4. D_{2n}-equivariant phase homology groups, where D_{2n} acts through rotations and reflections. These arise as homotopy groups of Anderson duals of $MTSpin \wedge X_n$ and $MTSpin^c \wedge X_n$, where X_n is one of $(BD_{2n})^{2-V_\lambda}$ or $(BD_{2n})^{\mathrm{Det}(V_\lambda)-1}$. See §4.4.4 for details and proofs.

Let λ be the standard real 2-dimensional representation of D_{2n} and $V_\lambda \to BD_{2n}$ be the associated vector bundle. Let s be a reflection in D_{2n} and r a rotation through the angle $2\pi/n$. Then, define $x, y \in H^1(BD_{2n}; \mathbb{Z}/2) = \mathrm{Hom}(D_{2n}, \mathbb{Z}/2)$ by

$$x(s^\ell r^m) \coloneqq \ell \bmod 2 \tag{4.4.17a}$$

$$y(s^\ell r^m) \coloneqq m \bmod 2. \tag{4.4.17b}$$

In the representation λ, $s^\ell r^m \in D_{2n}$ acts by an orientation-reversing endomorphism iff ℓ is odd, so $w_1(V_\lambda) = x$.

Proposition 4.4.18 ([**Sna13**, Theorem 4.6], [**Tei92**, §2.3], [**Han93**, Theorems 5.5 and 5.6])**.**

(1) If n is odd, $H^*(BD_{2n}; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]$.

(2) If $n \equiv 0 \bmod 4$, $H^*(BD_{2n}; \mathbb{Z}/2) \cong \mathbb{Z}/2[x, y, w]/(xy + y^2)$, where $|w| = 2$ and $w = w_2(V_\lambda)$.

(3) If $n \equiv 2 \bmod 4$, $H^*(BD_{2n}; \mathbb{Z}/2) \cong \mathbb{Z}/2[x, y]$.

Lemma 4.4.19. *For $n \equiv 2 \bmod 4$, $w_2(V_\lambda) = xy + y^2$.*

PROOF. Since $s, r^{n/2} \in D_{2n}$ commute, there is a map $j\colon \mathbb{Z}/2 \times \mathbb{Z}/2 \to D_{2n}$ sending $(1,0) \mapsto s$ and $(0,1) \mapsto r^{n/2}$. The pullback map $j^*\colon H^*(BD_{2n};\mathbb{Z}/2) \to H^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2;\mathbb{Z}/2)$ sends x and y to linearly independent elements of $H^1(B\mathbb{Z}/2 \times B\mathbb{Z}/2;\mathbb{Z}/2)$: one way to see this is to identify the pullback map with the map $\mathrm{Hom}(D_{2n},\mathbb{Z}/2) \to \mathrm{Hom}(\mathbb{Z}/2\times\mathbb{Z}/2,\mathbb{Z}/2)$ given by precomposing with j. Thus j^* is an isomorphism on $H^1(-;\mathbb{Z}/2)$. For both BD_{2n} and $B\mathbb{Z}/2 \times B\mathbb{Z}/2$, the mod 2 cohomology ring is the free symmetric algebra on $H^1(-;\mathbb{Z}/2)$, so j^* is an isomorphism of cohomology rings.

Thus we can compute $w_2(V_\lambda)$ by regarding λ as a $\mathbb{Z}/2\times\mathbb{Z}/2$ representation. Let $\ell_1 \subset \lambda$ be the fixed locus of s, which is a subspace, and ℓ_2 be its orthogonal complement. Then $\lambda = \ell_1 \oplus \ell_2$ as $(\mathbb{Z}/2\times\mathbb{Z}/2)$-representations. Both s and $r^{n/2}$ act nontrivially on ℓ_2; on ℓ_1, s acts trivially and $r^{n/2}$ acts nontrivially. Thus $w(\ell_1) = 1 + j^*(y)$, $w(\ell_2) = 1 + j^*(x) + j^*(y)$, and

$$w_2(j^*V_\lambda) = w_2(\ell_1) + w_1(\ell_1)w_1(\ell_2) + w_2(\ell_2) = j^*(y(x+y)). \tag{4.4.20}$$

$\square$

Lemma 4.4.21. *Suppose n is odd and $i\colon \mathbb{Z}/2 \hookrightarrow D_{2n}$ is the inclusion of $\langle s\rangle$. Let $V \to BD_{2n}$ be a virtual vector bundle such that $w_1(V)$, as an element $\mathrm{Hom}(D_{2n},\mathbb{Z}/2)$, is nonzero on s. Then, the induced map of Thom spectra $\widehat{\imath}\colon (B\mathbb{Z}/2)^{i^*V} \to (BD_{2n})^V$ is a 2-primary homotopy equivalence.*

PROOF. By the homology Whitehead theorem, it suffices to show $\widehat{\imath}$ induces an isomorphism on mod 2 cohomology. The Thom isomorphism rewords our question to be about the map $H^*(BD_{2n};\mathbb{Z}/2) \to H^*(B\mathbb{Z}/2;\mathbb{Z}/2)$, and Proposition 4.4.18 tells us that both $H^*(B\mathbb{Z}/2;\mathbb{Z}/2)$ and $H^*(BD_{2n};\mathbb{Z}/2)$ are abstractly isomorphic to $\mathbb{Z}/2[x]$ with $|x| = 1$; we will show $i^*x_{BD_{2n}} = x_{B\mathbb{Z}/2}$, implying i^* is a ring isomorphism. Since x is the only nonzero degree-one element and V and i^*V are both unorientable, $x = w_1(V)$ and $i^*x = w_1(i^*V) \neq 0$. $\square$

We will need the next calculations to determine the odd-primary torsion subgroups of the phase homology groups we calculate. Recall that $x \in H^1(BD_{2n};\mathbb{Z}/2)$ is equal to $w_1(V_\lambda)$.

Lemma 4.4.22 (Handel [**Han93**, Theorems 5.8, 5.9])**.**

$$H_*(BD_{2n};(\mathbb{Z}[1/2])_x) \cong \begin{cases} \mathbb{Z}/n, & n \equiv 1 \bmod 4 \\ 0, & \text{otherwise.} \end{cases} \tag{4.4.23}$$

Handel calculates $H^*(BD_{2n};\mathbb{Z}_x)$; use the universal coefficient theorem to switch to $\mathbb{Z}[1/2]$-homology.

Proposition 4.4.24. *Suppose $V \to BD_{2n}$ is a rank-zero virtual vector bundle with $w_1(V) = x$. Then the odd-torsion subgroup of $\widetilde\Omega_k^{\mathrm{Spin}}((BD_{2n})^V)$ is isomorphic to the odd-torsion subgroup of $\mathbb{Z}/n$ for $k = 1$, and vanishes for $k = 0, 2, 3,$ and 4.*

Proof. Apply the Atiyah-Hirzebruch spectral sequence for the completion of spin bordism at primes other than 2. Since $w_1(V) = x$, the Thom isomorphism identifies $\widetilde{H}_*((BD_{2n})^V) \cong H_*(BD_{2n}; \mathbb{Z}_x)$, and by Lemma 4.4.22 we know these groups away from 2. The only nonzero entry in the E^2-page of total degree less than 5 is $E^2_{1,0} \cong \mathbb{Z}/n$, so the spectral sequence collapses in the desired range and we conclude. □

Proposition 4.4.25. *With V as in Proposition 4.4.24, the odd-torsion subgroup of $\widetilde\Omega_k^{\mathrm{Spin}^c}((BD_{2n})^V)$ is isomorphic to the odd-torsion subgroup of $\mathbb{Z}/n$ for $k = 1$ and 3, and vanishes for $k = 0, 2,$ and 4.*

Proof. Use the Atiyah-Hirzebruch spectral sequence for the completion of $MTSpin^c$ at odd primes, just as for Proposition 4.4.24. □

4.4.4.1. *Class D, spinless case.* Since we are considering spinless fermions, the FCEP tells us to compute $[MTSpin \wedge (BD_{2n})^{2-V_\lambda}, \Sigma^{d+1} I_{\mathbb{Z}}]$.

Proposition 4.4.26. *For n odd, the first few spin bordism groups of X_n are*

$$\widetilde\Omega_0^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde\Omega_1^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/n$$
$$\widetilde\Omega_2^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde\Omega_3^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde\Omega_4^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/16,$$

and $\widetilde\Omega_5^{\mathrm{Spin}}(X_n)$ is torsion.

Proof. To compute the 2-torsion subgroups of these bordism groups, apply Lemma 4.4.21 with $2 - V_\lambda$ to get a 2-primary stable equivalence $(BD_{2n})^{2-V_\lambda} \simeq (B\mathbb{Z}/2)^{1-\sigma}$, then (4.2.10a) to get $MTSpin \wedge (B\mathbb{Z}/2)^{1-\sigma} \simeq MTPin^+$. Low-degree pin$^+$ bordism groups are calculated in [**Gia73b, KT90a, KT90b**]. For the odd-torsion subgroups, use Proposition 4.4.24. □

Now we turn to the case where $n \equiv 2 \bmod 4$.

Theorem 4.4.27. *When $n \equiv 2 \bmod 4$, the first few spin bordism groups of X_n are*

$$\widetilde{\Omega}_0^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/n$$
$$\widetilde{\Omega}_2^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde{\Omega}_3^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde{\Omega}_4^{\mathrm{Spin}}(X_n) \cong (\mathbb{Z}/2)^{\oplus 3},$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion.

As usual, this together with the universal property (1.1.50) of $I_{\mathbb{Z}}$ gives the $n \equiv 2 \bmod 4$ entries in Table 4.

Proof. We will use the Adams spectral sequence at the prime 2 to compute $\widetilde{\Omega}_d^{\mathrm{Spin}}(X_n)$ for $d \leq 7$. This only sees 2-primary information, but we already calculated the odd-torsion subgroup in Proposition 4.4.24. Recall that $w_1(V_\lambda) = x$ and (from Lemma 4.4.19) $w_2(V_\lambda) = xy + y^2$; thus $w_1(2 - V_\lambda) = x$ and $w_2(2 - V_\lambda) = x^2 + xy + y^2$. This tells us the Steenrod squares in $\widetilde{H}^*(X_n; \mathbb{Z}/2)$, e.g. $\mathrm{Sq}^1(U) = Ux$ and $\mathrm{Sq}^2(U) = U(x^2 + xy + y^2)$. Continuing in this vein determines the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X_n; \mathbb{Z}/2)$ in low degrees, as shown in Figure 7, left. We obtain a splitting as $\mathcal{A}(1)$-modules:

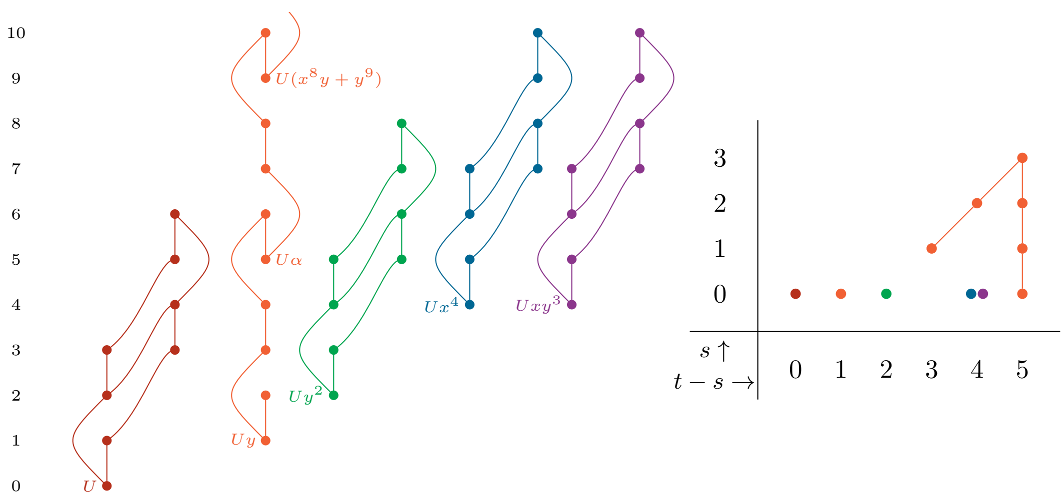


Figure 7. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BD_{2n})^{2-V_\lambda}; \mathbb{Z}/2)$ in low degrees, when $n \equiv 2 \bmod 4$. Here $\alpha := x^4y + y^5$. The submodule pictured here contains all elements of degree at most 5. Right: the E_2-page of the corresponding Adams spectral sequence computing ko-theory.

$$\widetilde{H}^*(X_n; \mathbb{Z}/2) \cong \mathcal{A}(1) \oplus \Sigma R_0 \oplus \Sigma^2\mathcal{A}(1) \oplus \Sigma^4\mathcal{A}(1) \oplus \Sigma^4\mathcal{A}(1) \oplus P. \tag{4.4.28}$$

The $\mathcal{A}(1)$-module R_0 is defined to be $\widetilde{H}^*((B\mathbb{Z}/2)^{1-\sigma};\mathbb{Z}/2)$; the copy appearing here is the indecomposable summand containing Uy. The submodule P contains no elements of degree below 6, so is irrelevant for our low-degree computations; we need to determine $\mathrm{Ext}(M)$ for the remaining summands. For $\Sigma^k\mathcal{A}(1)$, there is a single $\mathbb{Z}/2$ summand in topological degree k and filtration 0, and for ΣR_0, see [**GMM68**, §2] or [**BC18**, Figure 24]. Putting these together, we display the E_2-page of this Adams spectral sequence in Figure 7, right. In this range, a combination of h_1-equivariance and Margolis' theorem (Theorem 4.3.14) forces all differentials to vanish, and Margolis' theorem implies there are no hidden extensions, so we are done. □

Finally, consider the case that $n \equiv 0 \bmod 4$.

Theorem 4.4.29. *Let $n \equiv 0 \bmod 4$.*

$$\widetilde{\Omega}_0^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/n$$
$$\widetilde{\Omega}_2^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde{\Omega}_3^{\mathrm{Spin}}(X_n) \cong (\mathbb{Z}/2)^{\oplus 2}$$
$$\widetilde{\Omega}_4^{\mathrm{Spin}}(X_n) \cong (\mathbb{Z}/2)^{\oplus 4},$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion.

Proof of Theorem 4.4.29. First, by Proposition 4.4.24, the only odd-primary torsion in $\widetilde{\Omega}_k^{\mathrm{Spin}}(X_n)$ for $k \le 4$ is in degree 1. Draw the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = \widetilde{H}_p(X_n;\Omega_q^{\mathrm{Spin}}) \Longrightarrow \widetilde{\Omega}_{p+q}^{\mathrm{Spin}}(X_n). \tag{4.4.30}$$

After applying the Thom isomorphism, this needs as input $H_*(BD_{2n};\mathbb{Z}_x)$ and $H_*(BD_{2n};\mathbb{Z}/2)$. The former can be determined using Handel's calculation [**Han93**, Theorem 5.8] of $H^*(BD_{2n};\mathbb{Z}_x)$, and the latter can be determined from Proposition 4.4.18; in both cases use the universal coefficent theorem to pass from homology to cohomology. We obtain $E^2_{1,0} \cong \mathbb{Z}/n$ and $E^2_{0,1} \cong \mathbb{Z}/2$, so there are three options for $\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n)$: $\mathbb{Z}/n$, $\mathbb{Z}/n \oplus \mathbb{Z}/2$, or $\mathbb{Z}/2n$. We will address this ambiguity later.

Using Proposition 4.4.18, $w_1(2 - V_\lambda) = x$ and $w_2(2 - V_\lambda) = w + x^2$. Hence $\mathrm{Sq}^1(U) = Ux$ and $\mathrm{Sq}^2(U) = U(w + x^2)$. We also need the Steenrod squares of x, y, and w. For degree reasons, $\mathrm{Sq}(x) = x + x^2$ and $\mathrm{Sq}(y) = y + y^2$.

Lemma 4.4.31 ([**Mal11**, §4.1])**.** $\mathrm{Sq}(w) = w + wx + w^2$.

These and the Cartan formula determine the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X_n;\mathbb{Z}/2)$. In Figure 8, left, we display this structure in low degrees.

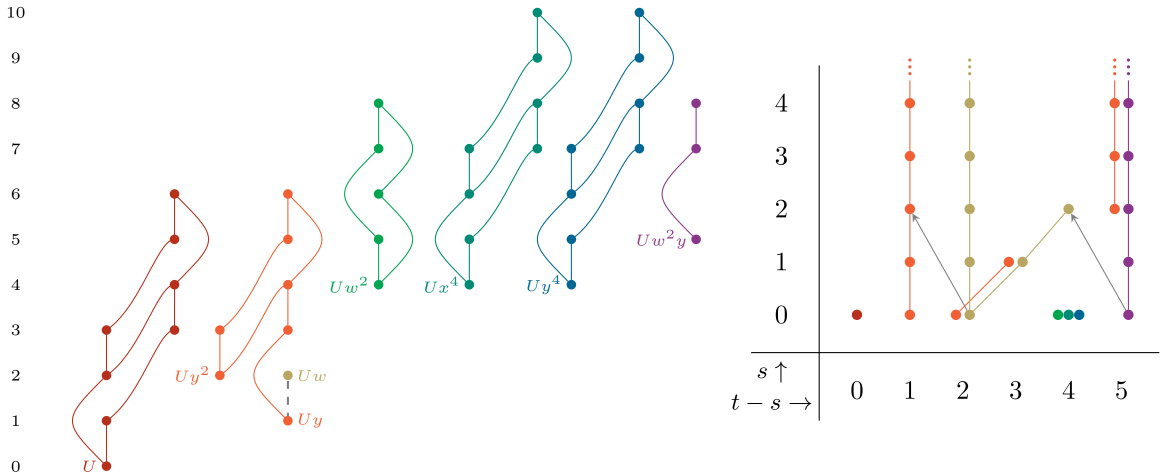


FIGURE 8. Left: the low-degree mod 2 cohomology of $(BD_{2n})^{2-V_\lambda}$ over $\mathcal{A}(1)$, $n \equiv 0 \bmod 4$. This summand contains all elements in degrees 5 and below. The dashed line indicates that the $\mathbb{Z}/2^r$ Bockstein maps Uy to Uw, which we need in Lemma 4.4.33. Right: the E_2-page of the Adams spectral sequence computing $\widetilde{ko}_*((BD_{2n})^{2-V_\lambda})^\wedge_2$. See Lemma 4.4.33 for how to address the differential in topological degree 2 and Lemma 4.4.36 to show the differential in topological degree 5 vanishes.

In particular,

$$\widetilde{H}^*(X_n;\mathbb{Z}/2) \cong \mathcal{A}(1) \oplus \Sigma R_2 \oplus \Sigma^2\mathbb{Z}/2 \oplus \Sigma^4 J \oplus \Sigma^4\mathcal{A}(1) \oplus \Sigma^4\mathcal{A}(1) \oplus \Sigma^5 \rotatebox[origin=c]{180}{?} \oplus P, \tag{4.4.32}$$

where P is 5-connected, and we define R_2, J, and $\rotatebox[origin=c]{180}{?}$ as follows. First, R_2 is defined to be the kernel of the augmentation map $\mathcal{A}(1) \to \mathbb{Z}/2$; the indecomposable summand in (4.4.32) isomorphic to ΣR_2 is generated by Uy and Uy^2. The *Joker* is the $\mathcal{A}(1)$-module $J := \mathcal{A}(1)/(\mathrm{Sq}^3)$; here it is generated by Uw^2. Finally, $\rotatebox[origin=c]{180}{?} := \mathcal{A}(1)/(\mathrm{Sq}^1, \mathrm{Sq}^2\mathrm{Sq}^3)$ and is called the *upside-down question mark*; here it is generated by Uw^2y. For each of these summands M in (4.4.32), $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(M, \mathbb{Z}/2)$ is known in the degrees relevant to us – except for P, which is too high-degree to affect our calculations anyways.

- For $\Sigma^k\mathcal{A}(1)$ there is a single $\mathbb{Z}/2$ in bidegree $s = 0$, $t = k$.
- For R_2, J, and $\rotatebox[origin=c]{180}{?}$, see [**BC18**, Figure 29].[18]
- For $\mathbb{Z}/2$, see [**BC18**, Figure 20].

Put these together to obtain the E_2-page as in Figure 8, right. Lemma 4.3.16 tells us the E_∞-page is torsion, so there must be nonzero differentials in the range shown, though not necessarily the d_2s pictured.

[18]The first calculations of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(R_2, \mathbb{Z}/2)$ and $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(J, \mathbb{Z}/2)$ that we know of are due to Adams-Priddy [**AP76**, §3].

The first nonzero differential is a d_r from the 2-line to the 1-line; by h_0-equivariance, it kills the entire yellow tower in the 2-line. Since a d_r differential decreases $t-s$ by 1 and increases s by r, on the E_{r+1}-page, the 2-line contains only the first r summands of the orange tower, and the 3-line contains only the orange $\mathbb{Z}/2$ summand in degree $s=0$. There can be no further differentials to or from the 1- or 2-lines, so we obtain $\mathbb{Z}/2^r$ in degree 1 and $\mathbb{Z}/2$ in degree 2.

Lemma 4.4.33. *2^r is the largest power of 2 dividing n, i.e. $\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/n$.*

Proof. The May-Milgram theorem [**MM81**] identifies Adams spectral sequence differentials between towers with Bockstein spectral sequence differentials. What it means here is that the lemma statement is equivalent to the statement that the Bockstein map $\beta\colon H^1(-;\mathbb{Z}/2^r) \to H^2(-;\mathbb{Z}/2)$ associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2^{r+1} \longrightarrow \mathbb{Z}/2^r \longrightarrow 0 \tag{4.4.34}$$

carries a preimage of Uy to Uw. Both of these classes are in the image of the pullback map induced by $(B\mathbb{Z}/n)^{2-V} \to (BD_{2n})^{2-V}$, and the Bockstein is natural with respect to the Thom isomorphism, so we just have to check this in the cohomology of $B\mathbb{Z}/n$, where it is true [**Cam17, DL20a**]. $\square$

The next differential that might be nonzero, and which is the only possibly nonzero differential to or from an element of degree 3 or 4, is $d_2\colon E_2^{0,5} \to E_2^{2,6}$. If this $d_2 = 0$, there is also an extension problem in degree $t-s=4$ of the form

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \widetilde{\Omega}_4^{\mathrm{Spin}}(X_n) \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow 0. \tag{4.4.35}$$

Lemma 4.4.36. *This d_2 vanishes, and the extension* (4.4.35) *splits.*

Proof. We will prove this by mapping to a simpler Adams spectral sequence that has already been studied, as depicted in Figure 9.

Because V_λ is the pullback of the tautological bundle $V_2 \to B\mathrm{O}_2$ along $B\lambda\colon BD_{2n} \to B\mathrm{O}_2$, we obtain a map of Thom spectra $f\colon X_n = (BD_{2n})^{2-V_\lambda} \to (B\mathrm{O}_2)^{2-V_2}$; the codomain is often denoted $\Sigma^2 MTO_2$. Under f, our $Uw \in \widetilde{H}^2(X_n;\mathbb{Z}/2)$ is the pullback of $Uw_2 \in \widetilde{H}^2(\Sigma^2 MTO_2)$.

The spin bordism of $\Sigma^2 MTO_2$ is identified with the bordism theory of the group $\mathrm{Pin}^{\tilde{c}+} := (\mathrm{Pin}^+ \ltimes \mathrm{Spin}_2)/\mu_2$. Invertible field theories for this tangential structure are believed to correspond to invertible topological phases of Altland-Zirnbauer type AII [**FH16a**, (9.25), (10.2)].[19]

[19]For further discussion, see also Metlitski [**Met15**] and Seiberg-Witten [**SW16**, §A.4].

Several authors study the Adams spectral sequence for $\Omega_*^{\mathrm{Pin}^{\tilde c+}} \cong \widetilde\Omega_*^{\mathrm{Spin}}(\Sigma^2 MTO_2)$ in low degrees, including Freed-Hopkins [**FH16a**, Figure 5, case $s=-2$], Campbell [**Cam17**, Example 6.10], and Wan-Wang-Zheng [**WWZ20**, §6.2.3]. Their work shows $Uw_2 \in \widetilde H^2(\Sigma^2 MTO_2;\mathbb Z/2)$ generates a $\Sigma^2\mathbb Z/2$ summand as an $\mathcal A(1)$-submodule of $\widetilde H^*(\Sigma^2 MTO_2;\mathbb Z/2)$, and therefore f^* restricts to an isomorphism from that $\Sigma^2\mathbb Z/2$ summand to our $\Sigma^2\mathbb Z/2$ summand generated by Uw. This means the submodule of the E_2-page for $\widetilde\Omega_*^{\mathrm{Spin}}(X_n)$ coming from $\Sigma^2\mathbb Z/2$ maps isomorphically onto the submodule of the E_2-page for $\widetilde\Omega_*^{\mathrm{Spin}}(\Sigma^2 MTO_2)$ coming from the $\Sigma^2\mathbb Z/2$ generated by Uw_2 — and crucially, in that spectral sequence, $E_2^{0,5}\cong 0$. See the commutative diagram of pink arrows in Figure 9. Thus the image of our d_2 under f vanishes, and the map between these spectral sequences on $E_2^{2,6}$s (the targets of these d_2s) is an isomorphism, so our d_2 also vanishes.

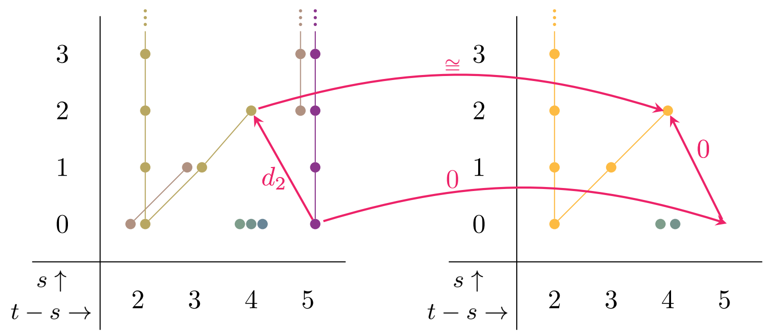


Figure 9. The map $X_n\to\Sigma^2 MTO_2$ induces a map between the Adams spectral sequences computing their *ko*-theory groups. We use this in Lemma 4.4.36 to show the pictured d_2 vanishes, as the square of pink arrows in the above figure is commutative. The right-hand side of this figure, which displays $\mathrm{Ext}(\widetilde H^*(\Sigma^2 MTO_2;\mathbb Z/2))$, is adapted from Campbell [**Cam17**, Figure 6.9].

Now suppose (4.4.35) does not split; then, there are elements $x,y\in\widetilde\Omega_4^{\mathrm{Spin}}(X_n)$ such that $x=2y$ and the image of x image in the E_∞-page of the Adams spectral sequence is the nonzero element of $E_\infty^{2,6}\cong\mathbb Z/2$. Then f maps this $\mathbb Z/2$ isomorphically onto a $\mathbb Z/2$ in the E_∞-page for $\Sigma^2 MTO_2$, so $f_*(x)\neq 0$. But $\Omega_4^{\mathrm{Pin}^{\tilde c+}}\cong(\mathbb Z/2)^{\oplus 3}$ [**FH16a**, Theorem 9.87], so no matter where y maps to, $2y=x\mapsto 0$, which is a problem. ☐

We have thus determined $\widetilde\Omega_d^{\mathrm{Spin}}(X_n)_2^\wedge$ for $d=3,4$, so we are done. ☐

4.4.4.2. *Class D, spin-1/2 case.*

Lemma 4.4.37. *V_λ is not pin$^-$.*

Proof. For n even, this follows by pulling back along $BC_n\to BD_{2n}$: we saw in §4.4.3 that the pullback is not spin, so V_λ cannot be pin$^-$. For n odd, pull back along the map $B\mathbb Z/2\to BD_{2n}$ induced by the inclusion of a reflection; the pullback is not pin$^-$, so neither is V_λ. ☐

Therefore by Theorem 4.2.11, we consider $X_n \coloneqq (BD_{2n})^{\mathrm{Det}(V_\lambda)-1}$.

Proposition 4.4.38. *For n odd, the first few spin bordism groups of X_n are*

$$\widetilde{\Omega}_0^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/2n$$
$$\widetilde{\Omega}_2^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/8$$
$$\widetilde{\Omega}_3^{\mathrm{Spin}}(X_n) \cong 0$$
$$\widetilde{\Omega}_4^{\mathrm{Spin}}(X_n) \cong 0,$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion.

Proof. To compute the 2-torsion subgroups of these bordism groups, apply Lemma 4.4.21 with $\mathrm{Det}(V_\lambda)-1$ get a 2-primary stable equivalence $(BD_{2n})^{Det(V_\lambda)-1} \simeq (B\mathbb{Z}/2)^{\sigma-1}$, then (4.2.10b) to get $MTSpin \wedge (B\mathbb{Z}/2)^{1-\sigma} \simeq MTPin^-$. Low-degree pin$^-$ bordism groups are calculated in [**ABP69, KT90b**]. For the odd-torsion subgroups, use Proposition 4.4.24. $\square$

Theorem 4.4.39. *When $n \equiv 2 \bmod 4$, the first few bordism groups of X_n are*

$$\widetilde{\Omega}_0^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/n \oplus \mathbb{Z}/2$$
$$\widetilde{\Omega}_2^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/4$$
$$\widetilde{\Omega}_3^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$
$$\widetilde{\Omega}_4^{\mathrm{Spin}}(X_n) \cong 0,$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion.

Proof. We establish a 2-primary equivalence $MTSpin \wedge X_n \simeq MTPin^- \wedge (B\mathbb{Z}/2)_+$, so the free and 2-torsion part of the spin bordism groups of X are isomorphic to the pin$^-$ bordism groups of $B\mathbb{Z}/2$. Once we finish this, we use work of Guo-Ohmori-Putrov-Wan-Wang [**GOP$^+$20**, §7.2.1] computing $\Omega_k^{\mathrm{Pin}^-}(B\mathbb{Z}/2)$ in degrees 5 and below to get the 2-primary part; for the odd-primary torsion, we use Proposition 4.4.24 as usual.

Lemma 4.4.40. *The inclusion $i\colon \mathbb{Z}/2\times\mathbb{Z}/2 \hookrightarrow D_{2n}$ given by a reflection and a half-turn induces a 2-primary equivalence of Thom spectra $(B(\mathbb{Z}/2\times\mathbb{Z}/2))^{i^*\mathrm{Det}(V_\lambda)-1} \xrightarrow{\simeq} (BD_{2n})^{\mathrm{Det}(V_\lambda)-1}$.*

Proof. The map $Bi\colon B(\mathbb{Z}/2\times\mathbb{Z}/2)\to BD_{2n}$ induces an equivalence on mod 2 cohomology, and therefore by the Thom isomorphism theorem also induces an equivalence on the mod 2 cohomology of the Thom spectra in question. This suffices by the stable Whitehead theorem. $\square$

The stable bundle $i^*\operatorname{Det}(V_\lambda)\to B(\mathbb{Z}/2\times\mathbb{Z}/2)$ splits as an exterior direct sum $\sigma\boxplus\underline{0}$, where $\sigma\to B\mathbb{Z}/2$ is the tautological line bundle. Therefore the Thom spectrum also splits: $(B(\mathbb{Z}/2\times\mathbb{Z}/2))^{i^*\operatorname{Det}(V_\lambda)-1}\simeq(B\mathbb{Z}/2)^{\sigma-1}\wedge(B\mathbb{Z}/2)_+$. Therefore by (4.2.10b),

$$MTSpin\wedge(BD_{2n})^{\operatorname{Det}(V_\lambda)-1}\simeq MTSpin\wedge(B\mathbb{Z}/2)^{\sigma-1}\wedge(B\mathbb{Z}/2)_+\simeq MTPin^-\wedge(B\mathbb{Z}/2)_+. \quad\square \tag{4.4.41}$$

Finally, let $n\equiv 0\bmod 4$. Recall $H^*(BD_{2n};\mathbb{Z}/2)\cong\mathbb{Z}/2[x,y,w]/(xy+y^2)$ with $|x|=|y|=1$ and $|w|=2$, so $\operatorname{Sq}(x)=x+x^2$ and $\operatorname{Sq}(y)=y+y^2$, and from Lemma 4.4.31, $\operatorname{Sq}(w)=w+wx+w^2$. The Stiefel-Whitney classes of $\operatorname{Det}(V_\lambda)$ tell us that if U is the Thom class, $\operatorname{Sq}^1(U)=Ux$ and $\operatorname{Sq}^2(U)=0$ in the cohomology of X_n.

Theorem 4.4.42. *For $n\equiv 0\bmod 4$, the first few bordism groups of X_n are*

$$\begin{aligned}\widetilde\Omega_0^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/2\\ \widetilde\Omega_1^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/n\oplus\mathbb{Z}/2\\ \widetilde\Omega_2^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/8\oplus\mathbb{Z}/4\\ \widetilde\Omega_3^{\mathrm{Spin}}(X_n)&\cong\mathbb{Z}/2\oplus\mathbb{Z}/2\\ \widetilde\Omega_4^{\mathrm{Spin}}(X_n)&\cong 0,\end{aligned}$$

and $\widetilde\Omega_5^{\mathrm{Spin}}(X_n)$ is torsion.

Proof. First, by Proposition 4.4.24, the only odd-primary torsion in $\widetilde\Omega_k^{\mathrm{Spin}}(X_n)$ for $k\le 4$ is in degree 1. Draw the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q}=\widetilde H_p(X_n;\Omega_q^{\mathrm{Spin}})\Longrightarrow\widetilde\Omega_{p+q}^{\mathrm{Spin}}(X). \tag{4.4.43}$$

After applying the Thom isomorphism, this needs as input $H_*(BD_{2n};\mathbb{Z}_x)$ and $H_*(BD_{2n};\mathbb{Z}/2)$. The former can be determined using Handel's calculation [Han93, Theorem 5.8] of $H^*(BD_{2n};\mathbb{Z}_x)$, and the latter can be determined from Proposition 4.4.18; in both cases use the universal coefficent theorem to pass from homology to cohomology. Since $E^2_{1,0}\cong\mathbb{Z}/n$ and $E^2_{0,1}\cong\mathbb{Z}/2$, there are three options for $\widetilde\Omega_1^{\mathrm{Spin}}(X_n)$: $\mathbb{Z}/n$, $\mathbb{Z}/n\oplus\mathbb{Z}/2$, or $\mathbb{Z}/2n$. We will learn which one is correct in our analysis of the 2-primary part below.

For the 2-primary part, we use the Adams spectral sequence as usual. By Lemma 4.3.22, a choice of a reflection in D_{2n} induces a splitting

$$X_n \xrightarrow{\simeq} (B\mathbb{Z}/2)^{\sigma-1} \vee M, \tag{4.4.44}$$

such that the map $\widetilde{H}^*(M;\mathbb{Z}/2) \to \widetilde{H}^*(X_n;\mathbb{Z}/2)$ is injective with image complementary to the subspace spanned by $\{Ux^i \mid i \geq 0\}$. We focus on $\mathit{MTSpin} \wedge M$, adding in the summands arising from $\mathit{MTSpin} \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq \mathit{MTPin}^-$ at the end. The $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ is determined by its image in $\widetilde{H}^*(X_n;\mathbb{Z}/2)$, which we know using Sq^1 and Sq^2 of x, y, w, and U via the Cartan formula. Using this, we draw this $\mathcal{A}(1)$-module structure in Figure 10, left.

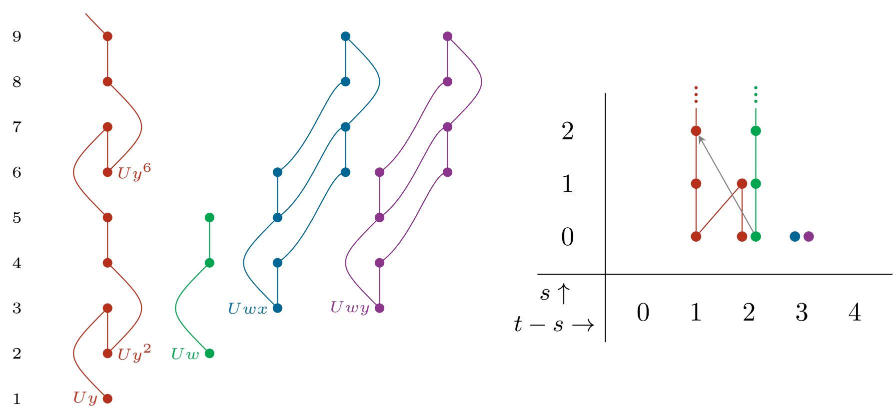


FIGURE 10. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M_n;\mathbb{Z}/2)$ in low degrees. The pictured summand contains all elements in degrees 4 and below. Right: the Ext of this module, which is the E_2-page of the Adams spectral sequence converging to $\widetilde{ko}_*(M_n)$. See the proof of Theorem 4.4.42 for more information.

As $\mathcal{A}(1)$-modules,

$$\widetilde{H}^*(M;\mathbb{Z}/2) \cong \Sigma R_1 \oplus \Sigma^2 \mathcal{\tilde{O}} \oplus \Sigma^3\mathcal{A}(1) \oplus \Sigma^3\mathcal{A}(1) \oplus P, \tag{4.4.45}$$

where P is 4-connected; we will see below that the 4-line is empty, so there are no nonzero differentials from $\mathrm{Ext}(P)$ to anything we care about. Here ΣR_1 is the indecomposable summand containing Uy. For the $\Sigma^k\mathcal{A}(1)$ summands, we know the Ext; for ΣR_1, see [**BC18**, Figure 26], and for $\Sigma^2\mathcal{\tilde{O}}$, see [**BC18**, Figure 29]. Assembling these, we display the E_2-page for $t-s \leq 4$ in Figure 10, right. Lemma 4.3.16 implies $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion, as claimed, and that there must be a differential d_r from the infinite tower in topological degree 2 to the infinite tower in topological degree 1, though it might not be the d_2 pictured.[20] Margolis' theorem and

[20]In fact, r is the largest number such that $2^r \mid n$. Like in the proof of Lemma 4.4.33, one can deduce this using the Bockstein from Uy to Uw and the May-Milgram theorem.

h_0-equivariance rule out any other nonzero differentials to or from elements with $t-s\le 4$. Therefore in this range, $E_{r+1}=E_\infty$. The infinite tower in topological degree 2 is killed by the differential, as are all but r of the $\mathbb{Z}/2$ summands of the infinite tower in topological degree 1. The first few 2-completed spin bordism groups of M_n are therefore $\mathbb{Z}/2^r$ in degree 1, $\mathbb{Z}/4$ in degree 2, $\mathbb{Z}/2\oplus\mathbb{Z}/2$ in degree 3, and 0 in degrees 0 and 4.

Finally, we add in the pin$^-$ bordism summands as computed in [**ABP69, KT90b**]: a $\mathbb{Z}/2$ in degrees 0 and 1, a $\mathbb{Z}/8$ in degree 2, and 0 otherwise. In particular, since the 2-torsion subgroup of $\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n)$ is of the form $\mathbb{Z}/2\oplus\mathbb{Z}/2^r$, $\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n)\cong\mathbb{Z}/n\oplus\mathbb{Z}/2$. ☐

4.4.4.3. *Class A, spinless case.* In this case, Theorem 4.2.24 asks us to consider $X_n := \mathit{MTSpin}^c\wedge(BD_{2n})^{2-V_\lambda}$.

Theorem 4.4.46. *For n odd, the first few spin bordism groups of X_n are*

$$\begin{aligned}\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/2\\ \widetilde{\Omega}_1^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/n\\ \widetilde{\Omega}_2^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/4\\ \widetilde{\Omega}_3^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/n\\ \widetilde{\Omega}_4^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/8\oplus\mathbb{Z}/2,\end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X_n)$ is torsion.

Proof. To compute the 2-torsion subgroups of these bordism groups, apply Lemma 4.4.21 with $2-V_\lambda$ to get a 2-primary stable equivalence $(BD_{2n})^{2-V_\lambda}\simeq(B\mathbb{Z}/2)^{1-\sigma}$, then (4.2.10c) to get $\mathit{MTSpin}^c\wedge(B\mathbb{Z}/2)^{1-\sigma}\simeq\mathit{MTPin}^c$. The pin^c bordism groups we need are calculated by Bahri-Gilkey [**BG87a, BG87b**]. For the odd-torsion subgroups, use Proposition 4.4.25. ☐

Theorem 4.4.47. *Let $n\equiv 2 \bmod 4$; then the low-degree spin^c bordism of X_n is*

$$\begin{aligned}\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/2\\ \widetilde{\Omega}_1^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/n\\ \widetilde{\Omega}_2^{\mathrm{Spin}^c}(X_n)&\cong(\mathbb{Z}/2)^{\oplus 2}\\ \widetilde{\Omega}_3^{\mathrm{Spin}^c}(X_n)&\cong\mathbb{Z}/2n\\ \widetilde{\Omega}_4^{\mathrm{Spin}^c}(X_n)&\cong(\mathbb{Z}/2)^{\oplus 4},\end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X_n)$ *is torsion.*

Proof. First, Proposition 4.4.25 computes the odd-torsion subgroups: a $\mathbb{Z}/n$ in degrees 1 and 3, and nothing else below degree 5.

To compute the 2-primary information we use the Adams spectral sequence over $\mathcal{E}(1)$, which converges to $\widetilde{ku}_*(X_n)$, together with Anderson-Brown-Peterson's isomorphism $\widetilde{\Omega}_n^{\mathrm{Spin}^c}(X_n) \xrightarrow{\cong} \widetilde{ku}_n(X_n) \oplus \widetilde{ku}_{n-4}(X_n)$ for $n \le 7$ [**ABP67**].

The $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X_n;\mathbb{Z}/2)$ that we calculated in (4.4.28) and displayed in Figure 7, left, determines the $\mathcal{E}(1)$-module structure: as $\mathcal{E}(1)$-modules, $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$. Therefore

$$\widetilde{H}^*(X_n;\mathbb{Z}/2) \cong \mathcal{E}(1) \oplus \Sigma R_0 \oplus \Sigma^2\mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1) \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^4\mathcal{E}(1) \oplus P, \tag{4.4.48}$$

where P is 5-connected; we draw a picture of this $\mathcal{E}(1)$-module in Figure 11, left.

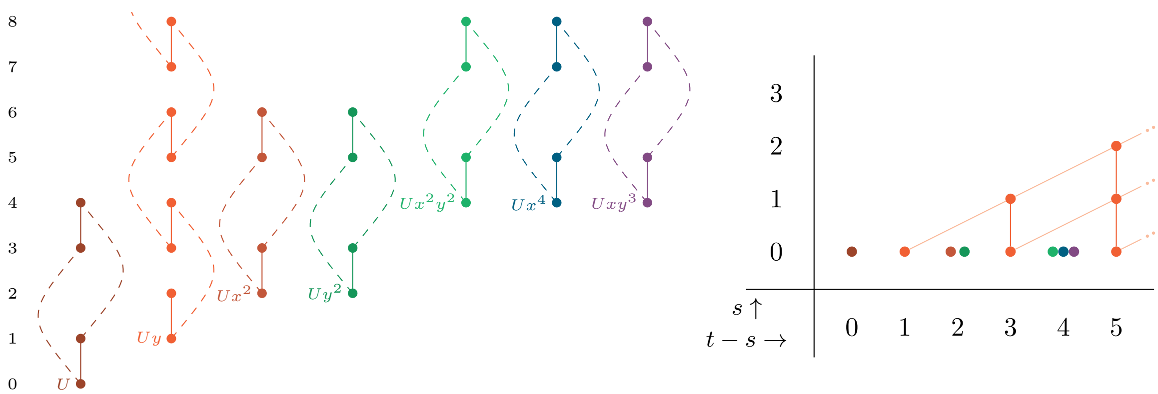


Figure 11. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*((BD_{2n})^{2-V_\lambda};\mathbb{Z}/2)$, $n \equiv 2 \bmod 4$, in low degrees. The pictured submodule contains all elements in degrees 5 and below. Right: the Adams E_2-page computing $\widetilde{ku}_*((BD_{2n})^{2-V_\lambda})$.

Next Ext. For $\Sigma^k\mathcal{E}(1)$, there is a unique $\mathbb{Z}/2$ summand, in degree $s = 0$, $t = k$; for ΣR_0, we must work a little harder.

Proposition 4.4.49. $\mathrm{Ext}_{\mathcal{E}(1)}^{s,t}(R_0,\mathbb{Z}/2)$ *is given in Figure 12, right.*

Proof. Our proof uses as input $\mathrm{Ext}_{\mathcal{E}(1)}(N_1)$, where N_1 is defined to be the $\mathcal{A}$-module $\Sigma^{-1}\widetilde{H}^*(\mathbb{RP}^2;\mathbb{Z}/2)$, with two $\mathbb{Z}/2$ summands connected by a Sq^1; this in turn defines its $\mathcal{A}(1)$- and $\mathcal{E}(1)$-module structures. Davis-Mahowald [**DM81**, §2] calculate $\mathrm{Ext}_{\mathcal{E}(1)}(N_1)$ as a graded vector space but we also need its $H^{*,*}(\mathcal{E}(1))$-module structure.

Let $\langle Q_1\rangle \subset \mathcal{E}(1)$ denote the subalgebra generated by Q_1, which is a two-dimensional vector space over $\mathbb{Z}/2$. As $\mathcal{E}(1)$-modules, $N_1 \cong \mathcal{E}(1) \otimes_{\langle Q_1\rangle} \mathbb{Z}/2$, so by the change-of-rings theorem (1.1.43), there are isomorphisms of

$H^{*,*}(\mathcal{E}(1))$-modules

$$\mathrm{Ext}_{\mathcal{E}(1)}(N_1) \cong \mathrm{Ext}_{\langle Q_1\rangle}(\mathbb{Z}/2) \cong \mathbb{Z}/2[v_1], \tag{4.4.50}$$

with $v_1 \in \mathrm{Ext}^{1,3}_{\mathcal{E}(1)}(N_1, \mathbb{Z}/2)$. The rightmost isomorphism in (4.4.50) uses Koszul duality [**BC18**, Remark 4.5.4], which applies because $\langle Q_1\rangle$ is an exterior algebra.

Now for R_0, we use the extension of $\mathcal{E}(1)$-modules

$$0 \longrightarrow \Sigma^2 R_0 \longrightarrow R_0 \longrightarrow N_1 \longrightarrow 0, \tag{4.4.51}$$

drawn in Figure 12, left.

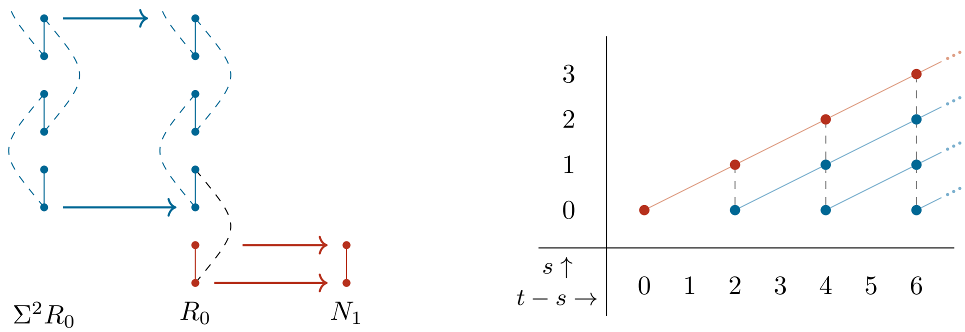


Figure 12. Left: the extension (4.4.51). Right: the long exact sequence it induces of Ext groups. See the proof of Proposition 4.4.49 for why the long exact sequence looks like this; the key feature is that there are no elements in odd topological degree, so all boundary maps vanish. The dashed lines are h_0-extensions which are not implied by the long exact sequence, but are shown in the proof of Proposition 4.4.49.

At first, all we know is $\mathrm{Ext}(N_1)$. Because this lives solely in even topological degrees, and $\Sigma^2 R_0$ is 2-connected, the long exact sequence diagram is empty in topological degree 1, so the boundary map

$$\delta\colon \mathrm{Ext}^{s,s+1}_{\mathcal{E}(1)}(R_0, \mathbb{Z}/2) \to \mathrm{Ext}^{s,s}_{\mathcal{E}(1)}(N_1, \mathbb{Z}/2) \tag{4.4.52}$$

vanishes, which tells us the line $t-s=0$ in $\mathrm{Ext}(R_0)$ consists of a single $\mathbb{Z}/2$ summand in filtration 0. Therefore the line $t-s=2$ in the long exact sequence diagram consists of two $\mathbb{Z}/2$ summands: one in filtration 1 coming from N_1, and one in filtration 0 coming from $\Sigma^2 R_0$. Since the 1-line of the diagram is empty and $\mathrm{Ext}(N_1)$ is concentrated in even degrees, the 3-line of the diagram is empty, so there are no differentials to the 2-line. Continuing in this way produces Figure 12, right.

Finally, acting by $h_0 \in H^{*,*}(\mathcal{E}(1))$ defines an isomorphism

$$\mathrm{Ext}^{0,2}_{\mathcal{E}(1)}(R_0, \mathbb{Z}/2) \to \mathrm{Ext}^{1,3}_{\mathcal{E}(1)}(R_0, \mathbb{Z}/2). \tag{4.4.53}$$

This can be checked directly from the definition: begin with the unique nontrivial map $R_0 \to \Sigma^2\mathbb{Z}/2$ and act on it by an extension representing h_0 (namely the extension $0 \to \Sigma\mathbb{Z}/2 \to N_1 \to \mathbb{Z}/2 \to 0$); the result is a nontrivial extension. $\square$

With $\mathrm{Ext}(\Sigma R_0)$ in hand, we return to our goal of computing $\widetilde{ku}_*(X_n)$. We draw the E_2-page of the Adams spectral sequence in (11), right. Margolis' theorem (Theorem 4.3.14) forces all differentials in this range to vanish, except possible differentials with target the 7-line, and there can be no hidden extensions in the range depicted. Thus for $n = 2k < 7$, $\widetilde{ku}_*(X_n) \cong (\mathbb{Z}/2)^{\oplus k+1}$ and for $n = 2k+1 < 8$, $\widetilde{ku}_*(X_n) \cong \mathbb{Z}/2^{k+1}$; we finish with the fact that the map $MTSpin^c \to ku \vee \Sigma^4 ku$ is 7-connected, so we can read off the spin^c bordism groups from the ku-homology groups. $\square$

Theorem 4.4.54. *If $n \equiv 0 \bmod 4$, write $n = 2^k m$ with m odd. The first few spin^c bordism groups of X_n are*

$$\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/n$$
$$\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X_n) \cong (\mathbb{Z}/2)^{\oplus 2}$$
$$\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/2n$$
$$\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X_n) \cong (\mathbb{Z}/2)^{\oplus 4},$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X_n)$ is torsion.

Proof of Theorem 4.4.54. By Proposition 4.4.25, the odd-primary torsion is isomorphic to the odd-primary torsion of $\mathbb{Z}/n$ in degrees 1 and 3 and vanishes in degrees 0, 2, and 4.

At 2, we use the Adams spectral sequence. We described the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X;\mathbb{Z}/2)$ in (4.4.32) and draw it in Figure 8; this determines the $\mathcal{E}(1)$-module structure, with isomorphisms of $\mathcal{E}(1)$-modules $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$, $R_2 \cong \mathring{\mathcal{O}} \oplus \Sigma\mathcal{E}(1)$ and $J \cong \mathcal{E}(1) \oplus \Sigma^2\mathbb{Z}/2$. Hence as $\mathcal{E}(1)$-modules,

$$\widetilde{H}^*(X;\mathbb{Z}/2) \cong \mathcal{E}(1) \oplus \Sigma\mathring{\mathcal{O}} \oplus \Sigma^2\mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1) \oplus \Sigma^2\mathbb{Z}/2 \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^5\mathring{\mathcal{O}} \oplus P, \tag{4.4.55}$$

where P is 5-connected. We draw this $\mathcal{E}(1)$-module in Figure 13, left.

We calculated $\mathrm{Ext}(\mathbb{Z}/2)$ in (1.1.45), and Adams-Priddy [**AP76**, §3] show

$$\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(\mathring{\mathcal{O}}, \mathbb{Z}/2) \cong \mathrm{Ext}^{s+1,t+1}_{\mathcal{E}(1)}(\mathbb{Z}/2,\mathbb{Z}/2), \tag{4.4.56}$$

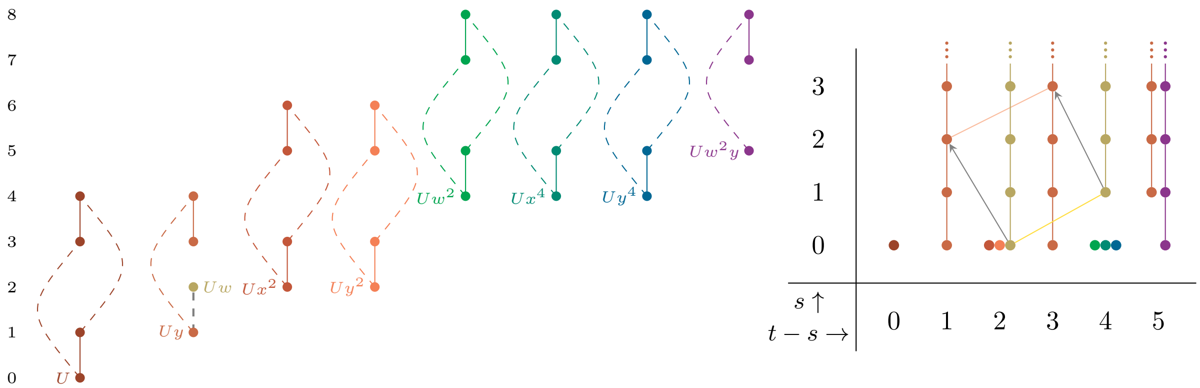


FIGURE 13. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*((BD_{2n})^{2-V_\lambda};\mathbb{Z}/2)$, $n \equiv 0 \bmod 4$, in low degrees. The pictured submodule contains all elements in degrees 5 and below. The gray dashed line indicates that the $\mathbb{Z}/2^r$ Bockstein maps a preimage of Uy to Uw, which we use in the proof of Theorem 4.4.54. Right: the E_2-page for the Adams spectral sequence computing $\widetilde{ku}_*((BD_{2n})^{2-V_\lambda})$. The two pictured differentials are related by a v_1-action.

with the isomorphism intertwining the $H^{*,*}(\mathcal{E}(1))$-actions. We can therefore draw the E_2-page of the Adams spectral sequence in Figure 13, right. We hide most v_1-actions to declutter the diagram.

The first differential that could be nonzero is from the 2-line to the 1-line; as differentials are h_0-equivariant, if a d_r differential is nonzero on one summand in the tower on the 2-line, then it is nonzero on the entire tower, so we refer to differentials between towers. The May-Milgram theorem [MM81] characterizes differentials between towers: there is a d_r differential between those two towers iff the Bockstein $\beta\colon H^1(-;\mathbb{Z}/2^r) \to H^2(-;\mathbb{Z}/2)$ carries a preimage of Uy to Uw. The Thom isomorphism is natural with respect to this Bockstein, so it suffices to know whether $\beta(y) = w$ in $H^2(BD_{2n};\mathbb{Z}/2)$, and we saw this in the proof of Lemma 4.4.33, where r is the largest number such that $2^r \mid n$. This means that the 2-torsion in $\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X_n)$ is isomorphic to that of $\mathbb{Z}/n$, so along with our odd-torsion computation we see that $\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/n$.

The other differential we need to resolve in range goes from the tower in the 4-line to the tower in the 3-line. Action by $v_1 \in ku_2$ carries the tower in the 2-line to the tower in the 4-line, and the tower in the 1-line to the tower in the 3-line, and differentials are v_1-equivariant, so there is also a d_r differential between these towers. As seen in Figure 13, right, on the E_∞-page there are $r+1$ $\mathbb{Z}/2$ summands on the 3-line, all connected, so together with our odd-torsion computation we see that $\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/2n$.

There can be no other nonzero differentials in range, and Margolis' theorem precludes any hidden extensions, so we are done. ∎

4.4.4.4. *Class A, spin-1/2 case.*

Lemma 4.4.57. *V_λ is pin^c iff n is odd.*

Proof. For n odd, we saw that inclusion of a reflection defines a map $B\mathbb{Z}/2 \to BD_{2n}$ which is an isomorphism on mod 2 cohomology. Therefore we can compute Stiefel-Whitney classes of V_λ by pulling back to $B\mathbb{Z}/2$, and we saw that the pullback bundle is stably equivalent to a line bundle, so $w_2 = 0$.

For n even, recall that V_λ is pin^c iff $\beta(w_2(V_\lambda)) = 0$, where $\beta\colon H^k(-;\mathbb{Z}/2) \to H^{k+1}(-;\mathbb{Z})$ is the integral Bockstein. Lemma 4.3.20 means it suffices to show $\mathrm{Sq}^1(w_2(V_\lambda)) \neq 0$. In the notation of Proposition 4.4.18, for $n \equiv 2 \bmod 4$, $w_2(V_\lambda) = xy + y^2$, and $\mathrm{Sq}^1(xy + y^2) = x^2y + xy^2 \neq 0$. For $n \equiv 0 \bmod 4$, $w_2(V_\lambda) = w$, and by Lemma 4.4.31, $\mathrm{Sq}^1(w) \neq 0$. $\square$

Therefore for n odd, we consider $X_n := (BD_{2n})^{2-V_\lambda}$. We computed $\Omega_k^{\mathrm{Spin}^c}(X_n)$ for $k \leq 4$ in Theorem 4.4.46.

For n even, Theorem 4.2.24 directs us to the spin^c bordism of $X_n := (BD_{2n})^{\mathrm{Det}(V_\lambda)-1}$.

Theorem 4.4.58. *If $n \equiv 2 \bmod 4$, the first few spin^c bordism groups of X_n are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}/n\\
\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}/4 \oplus \mathbb{Z}/2\\
\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}/n \oplus \mathbb{Z}/2\\
\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2.
\end{aligned}$$

Because Lemma 4.3.16 implies $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X_n)$ is torsion, the phase homology groups for this symmetry type are $\mathbb{Z}/n \oplus \mathbb{Z}/2$ for $d = 2$ and $\mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$ for $d = 3$.

The 2-local equivalence $\mathit{MTSpin} \wedge X_n \simeq \mathit{MTPin}^- \wedge (B\mathbb{Z}/2)_+$ we used in Theorem 4.4.39 implies a 2-local equivalence $\mathit{MTSpin}^c \wedge X_n \simeq \mathit{MTPin}^c \wedge (B\mathbb{Z}/2)_+$, so when $n = 2$, these are also the pin^c bordism groups of $\mathbb{Z}/2$. This may be of independent interest.

Proof. We can read the odd-primary torsion off of Proposition 4.4.25. For 2-primary torsion we use the Adams spectral sequence over $\mathcal{E}(1)$ as usual. Recall from the proof of Theorem 4.4.39 that $(BD_{2n})^{\mathrm{Det}(V_\lambda)-1} \simeq (B\mathbb{Z}/2)^{\sigma-1} \wedge (B\mathbb{Z}/2)_+$. Guo-Ohmori-Putrov-Wan-Wang [**GOP+20**, §7.2.1] determine the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1} \wedge B\mathbb{Z}/2;\mathbb{Z}/2)$ in low degrees. Using their work, and the isomorphisms of

$\mathcal{E}(1)$-modules $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \mathcal{E}(1)$ and $R_5 \cong \mathcal{E}(1) \oplus \Sigma R_0$, there is an isomorphism of $\mathcal{E}(1)$-modules

$$\widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1} \wedge B\mathbb{Z}/2; \mathbb{Z}/2) \cong \Sigma\mathcal{E}(1) \oplus \Sigma^2 R_0 \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus P, \tag{4.4.59}$$

where P is 4-connected. Since we began with $(B\mathbb{Z}/2)^{\sigma-1} \wedge (B\mathbb{Z}/2)_+$, this does not account for everything; the disjoint basepoint gives us another summand equivalent to

$$MTSpin^c \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq MTPin^c \tag{4.4.60}$$

by (4.2.10c). We will add in the pin^c bordism groups coming from this summand, which can be read off from the work of Bahri-Gilkey [**BG87a, BG87b**], after running the Adams spectral sequence for the other summand.

Returning to (4.4.59), we will see momentarily that $E_2^{s,t}$ is empty when $t - s = 4$ and $s \geq 2$, which precludes differentials from the 5-line to the 4-line and therefore means that P does not affect the calculations we make. In Figure 14, left, we draw (4.4.59). We computed $\mathrm{Ext}(R_0)$ in Proposition 4.4.49, so we can draw the E_2-page of the Adams spectral sequence for $\widetilde{ku}_*((B\mathbb{Z}/2)^{\sigma-1} \wedge B\mathbb{Z}/2)$, as in Figure 14, right. In the degrees we care about, this collapses, and we deduce the spin^c bordism of $(B\mathbb{Z}/2)^{\sigma-1} \wedge B\mathbb{Z}/2$ and combine it with pin^c bordism to conclude. $\square$

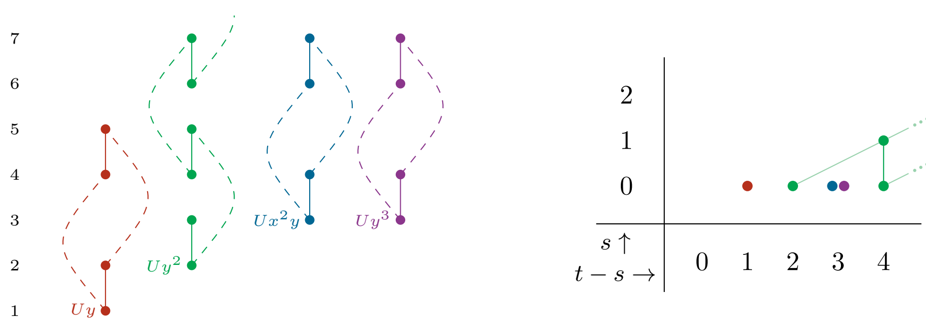


FIGURE 14. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1} \wedge B\mathbb{Z}/2; \mathbb{Z}/2)$ in low degrees. The pictured submodule contains all elements in degrees 4 and below. Right: Ext of this submodule, which is the E_2-page of the Adams spectral sequence computing $\widetilde{ku}_*(M_n)$ for $t - s \leq 4$. See the proof of Theorem 4.4.58 for more information.

Theorem 4.4.61. *If $n \equiv 0 \bmod 4$, the first few spin^c bordism groups of X_n are*

$$\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/2$$
$$\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/n$$
$$\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$$
$$\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/(n/2) \oplus (\mathbb{Z}/2)^{\oplus 2}$$
$$\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X_n) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2.$$

Because Lemma 4.3.16 implies $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X_n)$ is torsion, the phase homology groups for this symmetry type are $\mathbb{Z}/(n/2) \oplus (\mathbb{Z}/2)^{\oplus 2}$ for $d = 2$ and $\mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$ for $d = 3$.

Proof. We closely follow the proof of Theorem 4.4.42. For odd-primary torsion, use Proposition 4.4.25 to see that the odd-primary torsion in the range we care about is isomorphic to the odd torsion in $\mathbb{Z}/n$ in degrees 1 and 3, and is 0 in degrees 0, 2, and 4.

On to the prime 2. In Theorem 4.4.42, we established a splitting $X_n \simeq (B\mathbb{Z}/2)^{\sigma-1} \vee M_n$, allowing us to focus solely on $\widetilde{\Omega}_*^{\mathrm{Spin}}(M_n)$: $\mathit{MTSpin}^c \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq \mathit{MTPin}^c$ (4.2.10c), and we know pin^c bordism groups thanks to Bahri-Gilkey [**BG87a, BG87b**]. In (4.4.45), we determined the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M_n;\mathbb{Z}/2)$ in low degrees, and the isomorphisms of $\mathcal{E}(1)$-modules $R_1 \cong \mathbb{Z}/2 \oplus \Sigma R_0$ and $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$ mean that as $\mathcal{E}(1)$-modules,

$$\widetilde{H}^*(M_n;\mathbb{Z}/2) \cong \Sigma\mathbb{Z}/2 \oplus \Sigma^2 R_0 \oplus \Sigma^2 \mathcal{\mathring{O}} \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus P, \tag{4.4.62}$$

where P is 4-connected. A priori, $\mathrm{Ext}(P)$ could have nonzero differentials to elements of the 4-line, but we will see that this does not happen without needing to compute $\mathrm{Ext}(P)$. In Figure 15, left, we draw (4.4.62). To determine the E_2-page of the Adams spectral sequence, see (1.1.45) for $\mathrm{Ext}(\mathbb{Z}/2)$, Proposition 4.4.49 for $\mathrm{Ext}(R_0)$, and (4.4.56) for $\mathrm{Ext}(\mathcal{\mathring{O}})$. We draw the E_2-page of the Adams spectral sequence for $\widetilde{ku}_*(M_n)$, as in Figure 15, right — though for legibility, most v_1-actions are hidden. Lemma 4.3.16 implies there must be differentials in this range, though not necessarily the d_2s pictured.

For $\widetilde{\Omega}_1^{\mathrm{Spin}^c}(M_n)$ to be torsion, there must be a differential d_r from the 2-line to the 1-line; then, $\widetilde{\Omega}_1^{\mathrm{Spin}^c}(M_n) \cong \mathbb{Z}/2^r$, and since $\Omega_1^{\mathrm{Pin}^c} \cong 0$, $\widetilde{\Omega}_1^{\mathrm{Spin}^c}(M_n) \cong \mathbb{Z}/2^r$ as well. Differentials between towers, such as this d_r, are characterized by the May-Milgram theorem [**MM81**], and just as in the proof of Theorem 4.4.54, we conclude r is the largest natural number such that $2^r \mid n$. Combining this with our odd-torsion calculation, $\widetilde{\Omega}_1^{\mathrm{Spin}^c}(M_n) \cong \mathbb{Z}/n$.

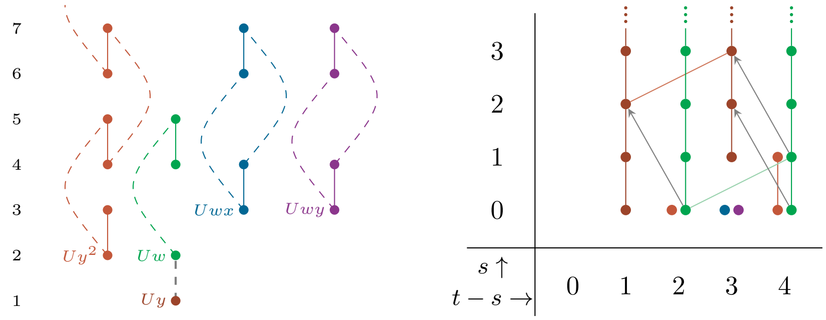


FIGURE 15. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M_n;\mathbb{Z}/2)$ in low degrees. The pictured submodule contains all elements in degrees 4 and below. The dashed line indicates a $\mathbb{Z}/2^r$ Bockstein, which we use to resolve a differential. Right: Ext of this submodule, which is the E_2-page of the Adams spectral sequence computing $\widetilde{ku}_*(M_n)$ for $t-s\le 4$. Most v_1-actions are hidden for readability. See the proof of Theorem 4.4.61 for more information.

Continuing in increasing topological degree, this d_r kills the entire orange tower in the 2-line, and we infer $\widetilde{\Omega}_2^{\mathrm{Spin}^c}(M_n)\cong\mathbb{Z}/2$. The green and blue summands in the 3-line survive and split off by Margolis' theorem. v_1-equivariance of differentials implies that $d_r\colon E_2^{s,4+s}\to E_2^{s+2,s+3}$ is nonzero, and again maps the orange tower to the dark red tower, leaving a single $\mathbb{Z}/2$ summand in $E_3^{1,4}$. There can be no further differentials to the 3-line, so $\widetilde{\Omega}_3^{\mathrm{Spin}^c}(M_n)^\wedge_2\cong\mathbb{Z}/2^{r-1}\oplus(\mathbb{Z}/2)^{\oplus 2}$. Our odd-primary calculation then tells us that $\widetilde{\Omega}_3^{\mathrm{Spin}^c}(M_n)\cong\mathbb{Z}/(n/2)\oplus(\mathbb{Z}/2)^{\oplus 2}$. Finally, the orange tower in the 4-line is killed by the d_r we most recently discussed, and the two light red $\mathbb{Z}/2$ summands in the 4-line cannot emit or receive differentials. Thus as promised $\mathrm{Ext}(P)$ does not have nonzero differentials to the 4-line, so we conclude by adding the pin^c bordism summands back in. $\square$

4.4.4.5. *Comparison with* [**ZWY$^+$20**]. Interacting fermionic phases equivariant for a dihedral group D_{2n} acting by rotations and reflections have also been studied by Zhang-Wang-Yang-Qi-Gu [**ZWY$^+$20**], who considered both spinless and spin-1/2 phases in dimension $2+1$ for all n, and in Altland-Zirnbauer class D. They also study systems without a spatial symmetry, using the extended supercohomology classification of Wang-Gu [**WG18, WG20**] to classify these phases and discuss the FCEP for dihedral groups. We find complete agreement with their results except for phases with spinless fermions when $n\equiv 0 \bmod 4$, where we predict $\mathbb{Z}/2\oplus\mathbb{Z}/2$ and they predict $\mathbb{Z}/2$. This appears to arise from a calculation error: as we note below in Remark 4.4.63, the comparison map between supercohomology and the Anderson dual of spin bordism is an isomorphism for this symmetry type.

Remark 4.4.63. The phases we classify are realized by the extended supercohomology classifications of Wang-Gu [**WG18, WG20**] and Kapustin-Thorngren [**KT17**].[21] Gaiotto-Johnson-Freyd [**GJF19**, §§5.4–5.6] determine that the extended supercohomology classification à la [**KT17, WG18**] is the cohomology of $(BD_{2n})^{2-V_\lambda}$ or $(BD_{2n})^{\mathrm{Det}(V_\lambda)-1}$ with respect to a spectrum they call $\mathrm{fGP}^\times_{\le 2}$, which is equivalent to the (-3)-connected cover of $I_\mathbb{Z}\mathit{MTSpin}$. Wang-Gu's refinement in [**WG20**] corresponds instead to the spectrum $\mathrm{fGP}^\times$, equivalent to the (-7)-connected cover of $I_\mathbb{Z}\mathit{MTSpin}$.[22]

The connective covering maps induce comparison maps from the classifications of fermionic phases using extended supercohomology to the classification of fermionic phases under our ansatz. For $\mathrm{fGP}^\times$, the map is sufficiently connected as to be an isomorphism between the classifications of $(d+1)$-dimensional phases for all $d \le 5$. For $\mathrm{fGP}^\times_{\le 2}$, the map is not always an isomorphism even for $d = 2$: the cokernel when computing supercohomology of X is $\widetilde{H}^0(X;\mathbb{Z})$, and this is nonzero e.g. for $X = (BC_n)^{2-V_\lambda}$ from §4.4.3. But for dihedral groups, $\widetilde{H}^0((BD_{2n})^\xi;\mathbb{Z})$ vanishes whenever $\xi \to BD_{2n}$ is a rank-0 unorientable virtual vector bundle, so in this case the comparison map is an isomorphism.

4.4.5. D_{2n} acting by rotations. The dihedral group D_{2n} can act on $\mathbb{R}^3$ in an orientation-preserving manner, where $C_n \subset D_{2n}$ acts by rotations in a plane and the remaining n elements act by rotations perpendicular to that plane. Said differently, this point group is defined by a representation $\lambda\colon D_{2n} \to \mathrm{SO}_3$ which decomposes as $\rho\oplus\sigma$, where ρ is the standard two-dimensional representation by rotations and reflections, and $\sigma\colon D_{2n} \to \mathrm{O}_1$ is the sign representation, which is the determinant of ρ. Confusingly, this point group is sometimes called "three-dimensional dihedral symmetry;" in this convention, the three-dimensional action by $\rho \oplus \underline{\mathbb{R}}$ is called *pyramidal symmetry*.

As far as we know, interacting fermionic phases for this D_{2n} symmetry have not been studied in the literature before.

For any representation $\phi\colon D_{2n} \to \mathrm{O}_d$, let $V_\phi \to BD_{2n}$ denote the associated vector bundle.

Lemma 4.4.64.

(1) If n is odd, V_λ is pin^c but not pin$^-$.

(2) If n is even, V_λ is not pin^c.

[21] These classifications concern phases with an internal D_{2n} symmetry, but the fermionic crystalline equivalence principle allows us to pass back and forth.

[22] The reader may at this point wonder why our classification is a generalized *homology* theory, while these extended supercohomology classifications are generalized *cohomology* theories. This is a subtle point. The passage between homology and cohomology occurs because in these dimensions, we may approximate *MTSpin* by *KO* due to Anderson-Brown-Peterson's [**ABP67**] study of the connectivity of the Atiyah-Bott-Shapiro map [**ABS64**], then use that *KO* is shifted Anderson self-dual [**And69, FMS07, HS14, Ric16, HLN20**] to pass between $I_\mathbb{Z}KO$-homology and $\Sigma^4 KO$-cohomology. See Freed-Hopkins [**FH19a**, §5.1] for further discussion.

n	Class D, spinless §4.4.5.1	Class D, spin-1/2 §4.4.5.2	Class A, spinless §4.4.5.3	Class A, spin-1/2 §4.4.5.4
$0 \bmod 4$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^{\oplus 2}$	0	$(\mathbb{Z}/2)^{\oplus 2}$
$2 \bmod 4$	0	$(\mathbb{Z}/2)^{\oplus 2}$	0	$(\mathbb{Z}/2)^{\oplus 3}$
$1, 3 \bmod 4$	0	0	0	0

Table 5. D_{2n}-equivariant phase homology groups, where D_{2n} acts faithfully on $\mathbb{R}^3$ by rotations. These arise as homotopy groups of Anderson duals of $MTSpin \wedge X_n$ and $MTSpin^c \wedge X_n$, where X_n is one of $(BD_{2n})^{3-V_\lambda}$ or $(BD_{2n})_+$. See §4.4.5 for details and proofs.

Proof. For (2), we show that if β is the integral Bockstein, $\beta w_2(V_\lambda) \neq 0$. By Lemma 4.3.20, it suffices to show $\mathrm{Sq}^1(w_2(V_\lambda)) \neq 0$. For $n \equiv 2 \bmod 4$,

$$w_2(V_\lambda) = w_2(\rho) + w_1(\rho)w_1(\sigma) + w_2(\sigma) = x^2 + xy + y^2, \tag{4.4.65a}$$

and $\mathrm{Sq}^1(x^2 + xy + y^2) = x^2y + xy^2$, and for $n \equiv 0 \bmod 4$,

$$w_2(V_\lambda) = w_2(\rho) + w_1(\rho)w_1(\sigma) + w_2(\sigma) = w + x^2, \tag{4.4.65b}$$

and $\mathrm{Sq}^1(w + x^2) = wx$, so in neither case is V_λ pin^c.

Now (1). Choose $i\colon \mathbb{Z}/2 \hookrightarrow D_{2n}$ given by a reflection; restricting to $\mathbb{Z}/2$, λ decomposes as $2\sigma \oplus \mathbb{R}$. Therefore $i^*V_\lambda \to B\mathbb{Z}/2$ is spin but not spin^c: $w_2(2\sigma) = w_1(\sigma)^2 = x^2$, and for any vector bundle V, $V \oplus V$ admits a complex structure, hence a spin^c structure. In particular, $\beta(w_2(i^*V_\lambda)) \neq 0$. The maps $\mathbb{Z}/2 \hookrightarrow D_{2n} \twoheadrightarrow \mathbb{Z}/2$ compose to the identity, so the induced maps on cohomology also compose to the identity. Therefore $\beta(w_2(V_\lambda)) \neq 0$ too. $\square$

These propositions are the analogues of Propositions 4.4.24 and 4.4.25, helping us calculate odd-primary torsion in phase homology groups.

Lemma 4.4.66 (Handel [**Han93**, Theorems 5.2, 5.3])**.**

$$\widetilde{H}_k(BD_{2n}; \mathbb{Z}[1/2]) \cong \begin{cases} \mathbb{Z}/n, & k \equiv 3 \bmod 4 \\ 0, & \text{otherwise.} \end{cases} \tag{4.4.67}$$

As usual, Handel computes $H^*(BD_{2n}; \mathbb{Z})$, and it is up to us to change to homology with $\mathbb{Z}[1/2]$ coefficients.

Proposition 4.4.68. *Suppose V is a rank-zero oriented virtual vector bundle.*

(1) The odd-torsion subgroup of $\widetilde{\Omega}_k^{\mathrm{Spin}}((BD_{2n})^V)$ is isomorphic to the odd-torsion subgroup of $\mathbb{Z}/n$ when $k = 3$ and vanishes for all other $k \leq 6$.

(2) *The odd-torsion subgroup of* $\widetilde{\Omega}_k^{\mathrm{Spin}^c}((BD_{2n})^V)$ *is isomorphic to the odd-torsion subgroup of* $\mathbb{Z}/n$ *when* $k=3$ *and* $k=5$ *and vanishes for all other* $k \le 6$.

Proof. It suffices to work at odd primes. There are odd-primary equivalences $\mathit{MTSpin} \to \mathit{MTSO}$ and $\mathit{MTSpin}^c \to \mathit{MTSO} \wedge (B\mathrm{U}_1)_+$; moreover, since V is oriented, there is a Thom isomorphism $\mathit{MTSO} \wedge (BD_{2n})_+ \xrightarrow{\simeq} \mathit{MTSO} \wedge (BD_{2n})^V$. Therefore it suffices to study $\widetilde{\Omega}_*^{\mathrm{SO}}(BD_{2n})$ for (1) and $\widetilde{\Omega}_*^{\mathrm{SO}}(BD_{2n} \wedge B\mathrm{U}_1)$ for (2) after completing at an odd prime p. Using Proposition 4.4.68 for input, as well as the Künneth formula to determine $H_*(BD_{2n} \wedge B\mathrm{U}_1)_p^\wedge$, one sees that the Atiyah-Hirzebruch spectral sequences computing these bordism groups collapse for degree reasons in total degree 6 and below. $\square$

4.4.5.1. *Class D, spinless case.* Let f_0^D denote the equivariant local system of symmetry types for this case. Theorem 4.2.11 tells us that to compute $Ph_*^{D_{2n}}(\mathbb{R}^3, f_0^D)$, we should study the spin bordism of $X_n \coloneqq (BD_n)^{3-V_\lambda}$.

Proposition 4.4.69. *Suppose* n *is odd. Then*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}}(X_n) &\cong \mathbb{Z}\\
\widetilde{\Omega}_1^{\mathrm{Spin}}(X_n) &\cong \mathbb{Z}/4\\
\widetilde{\Omega}_2^{\mathrm{Spin}}(X_n) &\cong 0\\
\widetilde{\Omega}_3^{\mathrm{Spin}}(X_n) &\cong \mathbb{Z}/n\\
\widetilde{\Omega}_4^{\mathrm{Spin}}(X_n) &\cong \mathbb{Z}\\
\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n) &\cong \mathbb{Z}/16\\
\widetilde{\Omega}_6^{\mathrm{Spin}}(X_n) &\cong 0,
\end{aligned}$$

and therefore $Ph_0^{D_{2n}}(\mathbb{R}^3, f_0^D) \cong 0$.

Proof. Proposition 4.4.68 shows that $\widetilde{\Omega}_k^{\mathrm{Spin}}(X_n)$ lacks odd-primary torsion for $k = 4, 5$, so it suffices to work at 2. The inclusion $\mathbb{Z}/2 \hookrightarrow D_{2n}$ induces an isomorphism $H^*(BD_{2n};\mathbb{Z}/2) \to H^*(B\mathbb{Z}/2;\mathbb{Z}/2)$, as we saw in the proof of Lemma 4.4.21, hence by naturality of the Thom isomorphism gives an isomorphism

$$\widetilde{H}^*(X_n;\mathbb{Z}/2) \xrightarrow{\cong} \widetilde{H}^*((B\mathbb{Z}/2)^{3-V_\lambda|_{B\mathbb{Z}/2}};\mathbb{Z}/2). \tag{4.4.70}$$

Restricted to $\mathbb{Z}/2$, $\lambda \cong 2\sigma \oplus \mathbb{R}$, so by the stable Whitehead theorem, (4.4.70) gives a stable 2-primary equivalence $X_n \simeq (B\mathbb{Z}/2)^{2-2\sigma}$. Campbell [**Cam17**, §7.8] computes $\widetilde{\Omega}_k^{\mathrm{Spin}}((B\mathbb{Z}/2)^{2-2\sigma})$, obtaining the free and 2-torsion summands we claim in the theorem statement.[23] □

Proposition 4.4.71 (Pedrotti [**Ped17**, Theorem 8.0.8])**.** *For $n \equiv 2 \bmod 4$, $\widetilde{\Omega}_4^{\mathrm{Spin}}(X_n) \cong \mathbb{Z}$, and by Lemma 4.3.16 $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion. Therefore $Ph_0^{D_{2n}}(\mathbb{R}^3, f_0^D)$ vanishes.*

Remark 4.4.72. Pedrotti reports this computation in terms of w_1 and w_2 of $3-V_\lambda$, rather than λ itself, so we should check that our characteristic classes agree with his: we want $w_1(3-V_\lambda) = 0$ and $w_2(3-V_\lambda) = x^2+xy+y^2$. Indeed $\mathrm{Im}(\lambda) \subset \mathrm{SO}_3$, so V_λ is orientable, and from (4.4.65a) that $w_2(V_\lambda) = x^2 + xy + y^2$. Since $w_1(V_\lambda) = 0$, these are also w_1 and w_2 of $3 - V_\lambda$, as desired.

Proposition 4.4.73 (Pedrotti [**Ped17**, Theorem 9.0.14])**.** *For $n \equiv 0 \bmod 4$, $\widetilde{\Omega}_4^{\mathrm{Spin}}(X_n) \cong \mathbb{Z} \oplus \mathbb{Z}/2$, and by Lemma 4.3.16 $\widetilde{\Omega}_5^{\mathrm{Spin}}(X_n)$ is torsion. Therefore $Ph_0^{D_{2n}}(\mathbb{R}^3, f_0^D) \cong \mathbb{Z}/2$.*

Pedrotti takes as input $w_1(3 - V_\lambda) = 0$ and $w_2(3 - V_\lambda) = w + x^2$, which agrees with the classes of V_λ (e.g. (4.4.65b)). Beware that what we call x he calls y, and vice versa!

4.4.5.2. *Class D, spin-1/2 case.* Let $f_{1/2}^D$ denote the equivariant local system of symmetry types for this case. Lemma 4.4.64 and Theorem 4.2.11 tell us that to compute $Ph_*^{D_{2n}}(\mathbb{R}^3, f_{1/2}^D)$, we should study the spin bordism of $(BD_n)^{\mathrm{Det}(V_\lambda)-1}$. Since V_λ is orientable, this is isomorphic to $\Omega_4^{\mathrm{Spin}}(BD_{2n})$.

Proposition 4.4.74. *Suppose n is odd. Then $\Omega_4^{\mathrm{Spin}}(BD_{2n}) \cong \mathbb{Z}$ and $\Omega_5^{\mathrm{Spin}}(BD_{2n}) \cong 0$, so $Ph_0^{D_{2n}}(\mathbb{R}^3, f_{1/2}^D) \cong 0$.*

Proof. The proof is almost the same as that of Proposition 4.4.69: by Proposition 4.4.68, there is no odd-primary torsion, and $B\mathbb{Z}/2 \to BD_{2n}$ induces an isomorphism on mod 2 cohomology, hence also on 2-local spin bordism, and Mahowald-Milgram [**MM76**] show $\Omega_4^{\mathrm{Spin}}(B\mathbb{Z}/2) \cong \mathbb{Z}$ and $\Omega_5^{\mathrm{Spin}}(B\mathbb{Z}/2) \cong 0$.[24] □

Proposition 4.4.75. *For n even, $\Omega_4^{\mathrm{Spin}}(BD_{2n}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^{\oplus 2}$.*

Proof. Pedrotti [**Ped17**, Theorems 8.0.4 and 9.0.3] shows $\Omega_4^{\mathrm{Spin}}(BD_{2n}) \cong \mathbb{Z} \oplus H_4(BD_{2n};\mathbb{Z})$, and the latter is computed by Handel [**Han93**, Theorem 5.2]. □

Bruner-Greenlees [**BG10**, Corollary 8.5.9] also compute this when n is a power of 2.

By Lemma 4.3.16, $\Omega_5^{\mathrm{Spin}}(BD_{2n})$ is torsion, so $Ph_*^D(\mathbb{R}^3, f_{1/2}^D) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

[23] Campbell computes only through dimension 5, and Beaudry-Campbell [**BC18**, Figure 26] shows how to extend Campbell's computation to dimension 6.

[24] This is also been computed by other methods by Mahowald [**Mah82**, Lemma 7.3], Bruner-Greenlees [**BG10**, Example 7.3.1], Siegemeyer [**Sie13**, Theorem 2.1.5], and García-Etxebarria and Montero [**GEM19**, (C.18)].

4.4.5.3. *Class A, spinless case.* Let f_0^A denote the equivariant local system of symmetry types in the spinless type A case. By Lemma 4.4.64, we should compute $\widetilde\Omega_*^{\mathrm{Spin}^c}(X_n)$, where $X_n \coloneqq (BD_{2n})^{3-V_\lambda}$.

When n is odd, V_λ is spin^c, so there is a Thom isomorphism $\mathit{MTSpin}^c \wedge X_n \simeq \mathit{MTSpin}^c \wedge (BD_{2n})_+$.

Theorem 4.4.76. *Suppose n is odd. Then*

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}\\
\Omega_1^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}/2\\
\Omega_2^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}\\
\Omega_3^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}/4n\\
\Omega_4^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}^2\\
\Omega_5^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}/8n \oplus \mathbb{Z}/2\\
\Omega_6^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}^2.
\end{aligned}$$

Therefore $Ph_0^{D_{2n}}(\mathbb{R}^3, f_0^A) \cong 0$.

Proof. Proposition 4.4.68 accounts for the odd-primary torsion, so we just have to work at 2. The map $\mathbb{Z}/2 \hookrightarrow D_{2n}$ induced by a choice of reflection defines an isomorphism on mod 2 cohomology, therefore by the stable Whitehead theorem is a 2-local stable equivalence. Therefore it defines an isomorphism $\Omega_*^{\mathrm{Spin}^c}(B\mathbb{Z}/2)_2^\wedge \to \Omega_*^{\mathrm{Spin}^c}(BD_{2n})_2^\wedge$, and the spin^c bordism of $B\mathbb{Z}/2$ is computed by Bahri-Gilkey [**BG87a, BG87b**]. ∎

Theorem 4.4.77. *Suppose $n \equiv 2 \bmod 4$. Then the first few spin^c bordism groups of X_n are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}\\
\widetilde\Omega_1^{\mathrm{Spin}^c}(X) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde\Omega_2^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}\\
\widetilde\Omega_3^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/n \oplus (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde\Omega_4^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}^2,
\end{aligned}$$

and $\widetilde\Omega_5^{\mathrm{Spin}^c}(X)$ is torsion.

Proof. The odd-torsion subgroups can be read off of (4.4.68). For the 2-primary part, we use the Adams spectral sequence over $\mathcal{E}(1)$. Letting U denote the Thom class, we saw $w_1(V_\lambda) = 0$, so $\mathrm{Sq}^1(U) = 0$, and (4.4.65a) $w_2(V_\lambda) = x^2 + xy + y^2$, so $\mathrm{Sq}^2(U) = U(x^2 + xy + y^2)$. Using this and the Cartan formula, we have an $\mathcal{E}(1)$-module isomorphism

$$\widetilde{H}^*(X_n; \mathbb{Z}/2) \cong \mathcal{O} \oplus \Sigma\mathcal{E}(1) \oplus \Sigma\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus P, \tag{4.4.78}$$

where P is 4-connected. We draw this in Figure 16, left. A priori $\mathrm{Ext}(P)$ could have nonzero differentials to the 4-line and therefore affect our computation, but we will see that this cannot happen without needing to determine $\mathrm{Ext}(P)$.

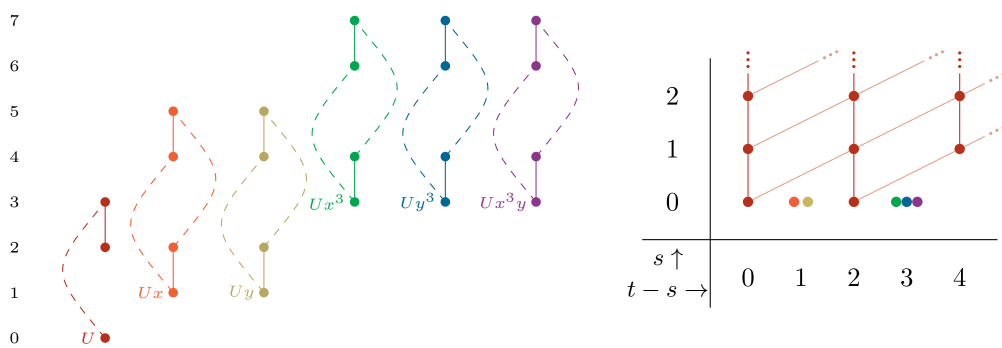


FIGURE 16. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(X_n; \mathbb{Z}/2)$ when $n \equiv 2 \bmod 4$. The pictured summand contains all elements in degrees 4 and below. Right: the E_2-page of the corresponding Adams spectral sequence computing $\widetilde{ku}_*(X_n)^\wedge_2$.

We calculated $\mathrm{Ext}_{\mathcal{E}(1)}(\mathcal{O})$ in (4.4.56), so we can draw the E_2-page of the Adams spectral sequence in Figure 16, right. h_0-equivariance rules out nonzero differentials in degrees 3 and below, but a priori there could be a nonzero differential from the 5-line to then 4-line. To rule this out, use Lemma 4.3.16 to see that $\widetilde{ku}_4(X_n)$ has one free summand. Therefore there cannot be any nonzero differentials to the 4-line: h_0-equivariance would mean that if there were such a differential, it would kill all but finitely many summands in the 4-line of the E_2-page, preventing $\widetilde{ku}_4(X_n)$ from having a free part. $\square$

Theorem 4.4.79. *When $n \equiv 0 \bmod 4$, the first few spin^c bordism groups of X_n are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}\\
\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X_n) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}\\
\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}/n \oplus (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X_n) &\cong \mathbb{Z}^2,
\end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X_n)$ *is torsion. Therefore* $Ph_0^{D_{2n}}(\mathbb{R}^3, f_0^A) \cong 0$.

Proof. The odd-torsion subgroups are calculated in Proposition 4.4.68. For the 2-torsion, we use the Adams spectral sequence over $\mathcal{E}(1)$. Recall that $w_1(V_\lambda) = 0$ and (from (4.4.65b)) $w_2(V_\lambda) = w + x^2$, so $w_1(3 - V_\lambda) = 0$ and $w_2(3 - V_\lambda) = w + x^2$. Thus in $\widetilde{H}^*(X_n; \mathbb{Z}/2)$, $\mathrm{Sq}^1(U) = 0$ and $\mathrm{Sq}^2(U) = U(w + x^2)$. Using this and the Cartan formula, we can compute the $\mathcal{E}(1)$-action on $\widetilde{H}^*(X_n; \mathbb{Z}/2)$, and find that

$$\widetilde{H}^*(X_n; \mathbb{Z}/2) \cong \mathcal{O} \oplus \Sigma\mathcal{E}(1) \oplus \Sigma\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathbb{Z}/2 \oplus \Sigma^4\mathcal{O} \oplus P, \tag{4.4.80}$$

where P is 4-connected. We draw this in Figure 17, left. We will see in a moment that $\mathrm{Ext}(P)$ has no nonzero differentials to elements in degree 4 and below, which means we can ignore it in our computations. We calculated $\mathrm{Ext}_{\mathcal{E}(1)}(\mathcal{O})$ in (4.4.56), so we can draw the E_2-page of the Adams spectral sequence in Figure 16, right. Margolis' theorem (Theorem 4.3.14) implies the only possible nonzero differentials from an element of

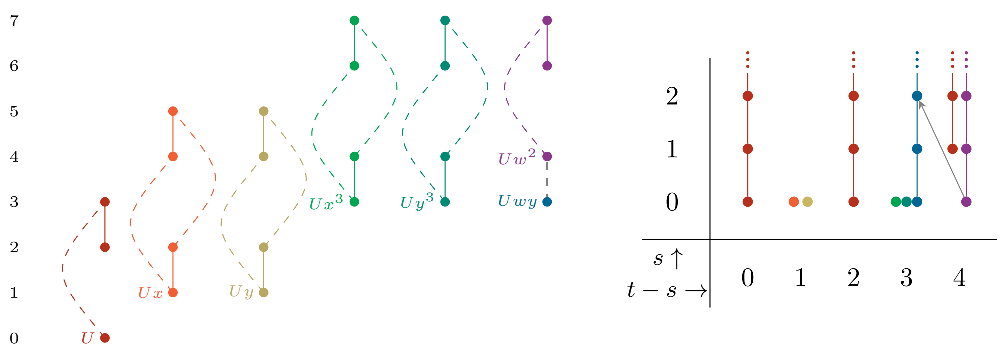


Figure 17. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(X_n; \mathbb{Z}/2)$ when $n \equiv 0 \bmod 4$. The pictured summand contains all elements in degrees 4 and below. The gray dashed line indicates a $\mathbb{Z}/2^r$ Bockstein, where r is the largest number for which $2^r \mid n$; this is not part of the $\mathcal{E}(1)$-module structure, but we use it in Theorem 4.4.79 to resolve a differential. Right: the E_2-page of the corresponding Adams spectral sequence computing $\widetilde{ku}_*(X_n)^\wedge_2$; v_1-actions are hidden for legibility. We will see in Theorem 4.4.79 that there is a d_r from the purple tower in the 4-line to the 3-line, though it is not always the d_2 pictured.

topological degree 4 or below are the differentials from a tower in the 4-line to the blue tower in the 3-line, and Lemma 4.3.16 implies $\widetilde{ku}_4(X_n)$ has free rank 1, so for some r this differential d_r is nonzero. Moreover, its source must be the purple tower: the red tower is in the image of $v_1\colon E_2^{s,s+2} \to E_2^{s+1,s+5}$, so if $d_r(x) = y$ for any element x of the red tower in degree 4, then y is also in the image of v_1, but the blue tower is not in this image. Therefore we know that d_r kills the entire purple tower in degree 4, and the red tower survives to the E_∞-page: the red tower supports no nonzero differentials to the 3-line, and if there were a differential from the 5-line to the red tower, h_0-linearity guarantees it would kill all but finitely many summands of the red tower, contradicting Lemma 4.3.16.

It remains only to determine the value of r. In $H^*(BD_{2n};\mathbb{Z}/2)$, the $\mathbb{Z}/2^k$ Bockstein carries (a preimage of) wy to w^2, where k is the largest number such that $2^k \mid n$. This can be checked by, e.g., pulling back to BC_n, where this Bockstein is discussed by [**Cam17, DL20a**]. The Thom isomorphism theorem implies the $\mathbb{Z}/2^k$ Bockstein sends (a preimage of) Uwy to Uw^2, and therefore by the May-Milgram theorem [**MM81**], $r = k$. $\square$

4.4.5.4. *Class A, spin-1/2 case.* Let $f^A_{1/2}$ denote the equivariant local system of symmetry types in the spin-1/2 type A case. In this case the ansatz tells us to study $\Omega_*^{\mathrm{Spin}^c}(BD_{2n})$.

Proposition 4.4.81. *For n odd, $Ph_0^{D_{2n}}(\mathbb{R}^3, f^A_{1/2}) = 0$.*

Proof. This follows from our computation of $\Omega_k^{\mathrm{Spin}^c}(BD_{2n})$ in Theorem 4.4.76. $\square$

Theorem 4.4.82. *Suppose $n \equiv 2 \bmod 4$. Then*

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}\\
\Omega_1^{\mathrm{Spin}^c}(BD_{2n}) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\Omega_2^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}\oplus\mathbb{Z}/2\\
\Omega_3^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}/n\oplus(\mathbb{Z}/4)^{\oplus 2}\oplus(\mathbb{Z}/2)^{\oplus 2}\\
\Omega_4^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}^2\oplus(\mathbb{Z}/2)^{\oplus 3}\\
\Omega_5^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}/n\oplus(\mathbb{Z}/8)^{\oplus 2}\oplus\mathbb{Z}/4\oplus(\mathbb{Z}/2)^{\oplus 5}\\
\Omega_6^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}^2\oplus(\mathbb{Z}/2)^{\oplus 6}.
\end{aligned}$$

Proof. We calculated the odd-primary torsion in these bordism groups in Proposition 4.4.68; now the 2-primary part. The inclusion $\mathbb{Z}/2\times\mathbb{Z}/2\to D_{2n}$ given by a reflection and a rotation by π induces an isomorphism on mod 2 cohomology, so by the stable Whitehead theorem, $BD_{2n}\to B(\mathbb{Z}/2\times\mathbb{Z}/2)$ is an equivalence after stabilizing and 2-completing. Therefore $\Omega_*^{\mathrm{Spin}^c}(BD_{2n})^\wedge_2 \xrightarrow{\cong} \Omega_*^{\mathrm{Spin}^c}(B(\mathbb{Z}/2\times\mathbb{Z}/2))^\wedge_2$; since $\mathit{MTSpin}^c\to ku\vee\Sigma^4 ku$ is an isomorphism in degrees 8 and below, it suffices to know $ku_*(B(\mathbb{Z}/2\times\mathbb{Z}/2))$.

Ossa [**Oss89**, Proposition 3] computes $ku_*(B(\mathbb{Z}/2\times\mathbb{Z}/2))$ by establishing an equivalence

$$ku\wedge B\mathbb{Z}/2\wedge B\mathbb{Z}/2\simeq(ku\wedge\Sigma^2 B\mathbb{Z}/2)\vee\Sigma^2 H(\mathbb{Z}/2[u,v]), \tag{4.4.83}$$

where the third term refers to a generalized Eilenberg-Mac Lane spectrum on the graded abelian group $\mathbb{Z}/2[u, v]$.[25] Using the stable splitting

$$\Sigma^\infty(B\mathbb{Z}/2 \times B\mathbb{Z}/2)_+ \simeq \mathbb{S} \vee \Sigma^\infty B\mathbb{Z}/2 \vee \Sigma^\infty B\mathbb{Z}/2 \vee \Sigma^\infty(B\mathbb{Z}/2 \wedge B\mathbb{Z}/2), \tag{4.4.84}$$

we see that $ku_*(B(\mathbb{Z}/2 \times \mathbb{Z}/2))$ can be assembled from the following pieces.

(1) $ku_*(\mathrm{pt})$, which contributes $\mathbb{Z}$ in even degrees and 0 in odd degrees.
(2) Two copies of $\widetilde{ku}_*(B\mathbb{Z}/2)$. Hashimoto [**Has83**, Theorem 3.1] shows each copy vanishes in even degrees and is isomorphic to $\mathbb{Z}/2^{k+1}$ in odd degree $2k+1$.
(3) $\widetilde{ku}_*(\Sigma^2 B\mathbb{Z}/2)$. Hashimoto (*ibid.*) shows this vanishes in even degrees and is isomorphic to $\mathbb{Z}/2^k$ in odd degree $2k+1$.
(4) $\pi_*(\Sigma^2 H\mathbb{Z}/2[u, v])$, which contributes 0 in degrees 0 and 1 and $(\mathbb{Z}/2)^{\oplus(k-1)}$ in degrees $k \geq 2$.

Putting this together and adding in the odd-primary torsion, we obtain $ku_k(BD_{2n})$ for $k \leq 6$; using the Anderson-Brown-Peterson isomorphism $\Omega_k^{\mathrm{Spin}^c}(BD_{2n}) \cong ku_k(BD_{2n}) \oplus ku_{k-4}(BD_{2n})$, valid for $k < 8$, we obtain the bordism groups in the theorem statement. ☐

Theorem 4.4.85. *Suppose $n \equiv 0 \bmod 4$. Then*

$$\begin{aligned}
\Omega_0^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}\\
\Omega_1^{\mathrm{Spin}^c}(BD_{2n}) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\Omega_2^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z} \oplus \mathbb{Z}/2\\
\Omega_3^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}/n \oplus (\mathbb{Z}/4)^{\oplus 2}\\
\Omega_4^{\mathrm{Spin}^c}(BD_{2n}) &\cong \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^{\oplus 2},
\end{aligned}$$

and $\Omega_5^{\mathrm{Spin}^c}(BD_{2n})$ is torsion. Therefore $Ph_0^{D_{2n}}(\mathbb{R}^3, f_{1/2}^A) \cong (\mathbb{Z}/2)^{\oplus 2}$.

Proof. Since $\Omega_*^{\mathrm{Spin}^c}(BD_{2n}) \cong \Omega_*^{\mathrm{Spin}^c}(\mathrm{pt}) \oplus \widetilde{\Omega}_*^{\mathrm{Spin}^c}(BD_{2n})$, we will focus on $\widetilde{\Omega}_*^{\mathrm{Spin}^c}(BD_{2n})$, adding on $\Omega_*^{\mathrm{Spin}^c}(\mathrm{pt})$ at the end. We also focus on the 2-primary story: the odd-primary torsion is calculated in Proposition 4.4.68.

Recall from Proposition 4.4.18 that $H^*(BD_{2n}; \mathbb{Z}/2) \cong \mathbb{Z}/2[x, y, w]/(xy + y^2)$, with $|x| = |y| = 1$ and $|w| = 2$. A choice of reflection induces a section of $BD_{2n} \to B\mathbb{Z}/2$, and therefore there is a spectrum M_n and

[25] Ossa's splitting (4.4.83) or its analogue on homotopy groups has also been proven in several other ways: see Johnson-Wilson [**JW97**], Bruner [**Bru99**, Corollary 3.3], Bruner-Greenlees [**BG03**, Example 4.11.2], Powell [**Pow14**], and Bruner-Mira-Stanley-Snaith [**BMSS15**, Theorem 2.12].

a splitting

$$\Sigma^\infty BD_{2n} \xrightarrow{\simeq} M_n \vee \Sigma^\infty B\mathbb{Z}/2, \tag{4.4.86}$$

such that as a subspace of $\widetilde{H}^*(BD_{2n};\mathbb{Z}/2)$, $\widetilde{H}^*(M_n;\mathbb{Z}/2)$ is complementary to the subspace S spanned by $\{x^n \mid n \geq 0\}$, because S is the image of the pullback map $\widetilde{H}^*(B\mathbb{Z}/2;\mathbb{Z}/2) \to \widetilde{H}^*(BD_{2n};\mathbb{Z}/2)$. Bahri-Gilkey [**BG87a, BG87b**] show that $\widetilde{\Omega}_4^{\mathrm{Spin}^c}(B\mathbb{Z}/2) \cong 0$, so we just have to understand $\widetilde{\Omega}_4^{\mathrm{Spin}^c}(M_n)$.

We will use the Adams spectral sequence over $\mathcal{E}(1)$ to show that $\widetilde{ku}_0(M_n) \cong 0$ and $\widetilde{ku}_4(M_n) \cong (\mathbb{Z}/2)^{\oplus 2}$, which suffices to prove the theorem. For degree reasons, $\mathrm{Sq}(x) = x + x^2$ and $\mathrm{Sq}(y) = y + y^2$, and in Lemma 4.4.31 we saw $\mathrm{Sq}(w) = w + wx + w^2$. Using this, we find that as $\mathcal{E}(1)$-modules,

$$\widetilde{H}^*(M_n;\mathbb{Z}/2) \cong \Sigma R_0 \oplus \Sigma^2\mathcal{E}(1) \oplus \Sigma^3\mathring{\mathcal{O}} \oplus \Sigma^4\mathbb{Z}/2 \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^4\mathcal{E}(1) \oplus P, \tag{4.4.87}$$

where P is 5-connected, and therefore too highly connected to affect our calculations. We draw this in Figure 18, left. We have already computed $\mathrm{Ext}(M)$ for the remaining summands M: see Proposition 4.4.49 for $\mathrm{Ext}(R_0)$, (4.4.56) for $\mathrm{Ext}(\mathring{\mathcal{O}})$, and (1.1.45) for $\mathrm{Ext}(\mathbb{Z}/2)$. Therefore we obtain the E_2-page of the Adams spectral sequence in Figure 18, right.

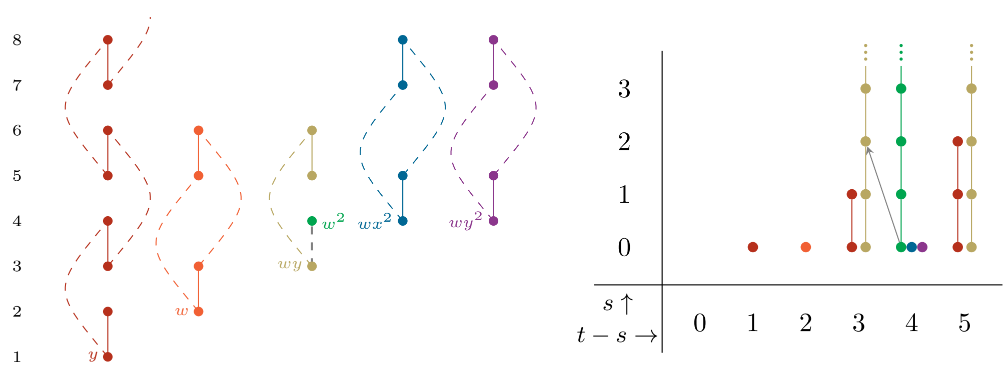


FIGURE 18. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M_n;\mathbb{Z}/2)$ in low degrees. The gray dashed line indicates a $\mathbb{Z}/2^r$ Bockstein, where r is the largest number such that $2^r \mid n$; this is not part of the $\mathcal{E}(1)$-module structure, but we will use it in Theorem 4.4.85 to resolve a differential. Right: the E_2-page of the Adams spectral sequence computing $\widetilde{ku}(M_n)^\wedge_2$. v_1-actions are hidden for readability. We will see in Theorem 4.4.85 that there is a nonzero differential from the 4-line to the 3-line, though it is not necessarily the d_2 pictured.

The 0-line is empty, so $\widetilde{ku}_0(M) \cong 0$, as promised. Lemma 4.3.16 implies $\widetilde{ku}_3(M_n)$ is torsion; therefore there must be a nonzero differential d_r from the purple tower in the 4-line to the yellow tower in the 3-line. As in previous examples (Lemma 4.4.33 and Theorems 4.4.54 and 4.4.61), k is the largest number such that $2^k \mid n$: the $\mathbb{Z}/2^k$ Bockstein sends a preimage of wy to w^2, which can be checked after pulling back to BC_n as usual. The May-Milgram theorem [**MM81**] then identifies $r = k$. Therefore from the E_{r+1}-page onward, the green tower is gone, and the 4-line consists only of the two $\mathbb{Z}/2$ summands in Adams filtration zero, so $\widetilde{ku}_4(M_n) \cong (\mathbb{Z}/2)^{\oplus 2}$. □

4.5. Examples: tetrahedral, octahedral, and icosahedral symmetries

Point group	Ref.	D, sp.-0	D, sp.-1/2	A, sp.-0	A, sp.-1/2
Chiral tet. $(A_4,\, T)$	§4.5.1	0	0	0	0
Pyrit. $(A_4 \times \mathbb{Z}_2,\, T_h)$	§4.5.2	$(\mathbb{Z}_2)^{\oplus 3}$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus (\mathbb{Z}_2)^{\oplus 3}$	$\mathbb{Z}_8 \oplus (\mathbb{Z}_2)^{\oplus 3}$
Full tet. $(S_4,\, T_d)$	§4.5.3	$\mathbb{Z}_4 \oplus (\mathbb{Z}_2)^{\oplus 2}$	0	$(\mathbb{Z}_2)^{\oplus 4}$	$\mathbb{Z}_8 \oplus (\mathbb{Z}_2)^{\oplus 2}$
Chiral oct. $(S_4,\, O)$	§4.5.4	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
Full oct. $(S_4 \times \mathbb{Z}_2,\, O_h)$	§4.5.5	$(\mathbb{Z}_2)^{\oplus 4}$	$(\mathbb{Z}_2)^{\oplus 2}$	$\mathbb{Z}_4 \oplus (\mathbb{Z}_2)^{\oplus 4}$	$\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus (\mathbb{Z}_2)^{\oplus 4}$
Chiral icos. $(A_5,\, I)$	§4.5.6	0	0	0	0
Full icos. $(A_5 \times \mathbb{Z}_2,\, I_h)$	§4.5.7	$(\mathbb{Z}_2)^{\oplus 3}$	$\mathbb{Z}_2$	$\mathbb{Z}_4 \oplus (\mathbb{Z}_2)^{\oplus 3}$	$\mathbb{Z}_8 \oplus (\mathbb{Z}_2)^{\oplus 3}$

TABLE 6. Phase homology groups in dimension $3+1$ equivariant with respect to various tetrahedral, octahedral, and icosahedral symmetries and the ways they can mix with fermion parity. In this table alone, $\mathbb{Z}_n$ denotes the cyclic group we usually call $\mathbb{Z}/n$, not the n-adic integers. See the referenced sections for how the fermionic crystalline equivalence principle associates this data with symmetry types for invertible TFTs.

4.5.1. Chiral tetrahedral symmetry. We compute phase homology groups equivariant for a chiral tetrahedral symmetry $\lambda\colon A_4 \to \mathrm{SO}_3$. As far as we know, this point group has not yet been considered by physicists in the setting of fermionic phases. We will show that our ansatz implies there are no nontrivial phases with either spinless or spin-1/2 fermions in both class D and class A. As usual, $V_\lambda \to BA_4$ denotes the vector bundle associated to λ.

Proposition 4.5.1. *$H^*(BA_4;\mathbb{Z}/2) \cong \mathbb{Z}/2[u,v,w]/(u^3 + v^2 + w^2 + vw)$, where $|u| = 2$ and $|v| = |w| = 3$. $\mathrm{Sq}(u) = u + v + w + u^2$, $\mathrm{Sq}(v) = v + u^2 + uw + v^2$, and $\mathrm{Sq}(w) = w + u^2 + uv + w^2$.*

Except for the Steenrod operations, this result can be found in several places, such as [**Kin**] and [**AM04**, Theorem III.1.3], so we will be brief.

PROOF SKETCH. Use the Lyndon-Hochschild-Serre spectral sequence [**Lyn48, Ser50, HS53**] for the short exact sequence $1 \to \mathbb{Z}/2 \times \mathbb{Z}/2 \to A_4 \to \mathbb{Z}/3 \to 1$; the mod 2 cohomology of $\mathbb{Z}/3$ is trivial, so the spectral sequence collapses, and

$$H^*(BA_4;\mathbb{Z}/2) \cong H^0(B\mathbb{Z}/3; H^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2;\mathbb{Z}/2)) = H^*(B\mathbb{Z}/2 \times B\mathbb{Z}/2;\mathbb{Z}/2)^{\mathbb{Z}/3}. \tag{4.5.2}$$

We can choose this $\mathbb{Z}/3$-action to be such that a generator of $\mathbb{Z}/3$ acts on $\mathbb{Z}/2\times\mathbb{Z}/2 = \{1,\alpha,\beta,\alpha+\beta\}$ by $\alpha\mapsto\alpha+\beta$, $\beta\mapsto\alpha$, and $\alpha+\beta\mapsto\beta$. In a mild abuse of notation, we identify $\mathbb{Z}/2\times\mathbb{Z}/2$ with $H^1(B\mathbb{Z}/2\times B\mathbb{Z}/2;\mathbb{Z}/2)\cong\mathrm{Hom}(\mathbb{Z}/2\times\mathbb{Z}/2,\mathbb{Z}/2)$: these are dual $\mathbb{Z}/2$-vector spaces, and we have a basis for one, which we identify with the dual basis vectors of the other. Thus $H^*(B\mathbb{Z}/2\times B\mathbb{Z}/2;\mathbb{Z}/2)\cong\mathbb{Z}/2[\alpha,\beta]$.

The unique nonzero degree-2 cohomology class fixed by $\mathbb{Z}/3$ is $u := \alpha^2+\alpha\beta+\beta^2$, and two linearly independent degree-3 classes fixed by $\mathbb{Z}/3$ are $v := \alpha^3+\alpha^2\beta+\beta^3$ and $w := \alpha^3+\alpha\beta^2+\beta^3$, whence the relation.

For the Steenrod squares, the identification in (4.5.2) of $H^*(BA_4;\mathbb{Z}/2)$ as a subalgebra of $H^*(B\mathbb{Z}/2\times B\mathbb{Z}/2;\mathbb{Z}/2)$ is the pullback map for $B\mathbb{Z}/2\times B\mathbb{Z}/2\to BA_4$, hence $\mathcal{A}$-equivariant, so we can compute $\mathrm{Sq}(u)$ in $H^*(B\mathbb{Z}/2\times B\mathbb{Z}/2;\mathbb{Z}/2)$; the computation follows from $\mathrm{Sq}(\alpha)=\alpha+\alpha^2$ and $\mathrm{Sq}(\beta)=\beta+\beta^2$. □

Lemma 4.5.3. *$w_1(V_\lambda)=0$ and $w_2(V_\lambda)=u$.*

Proof. Since V_λ is orientable, $w_1(V_\lambda)=0$, and since V_λ is not spin, $w_2(V_\lambda)\neq 0$. Since $H^2(BA_4;\mathbb{Z}/2)\cong\mathbb{Z}/2\cdot u$, $w_2(V_\lambda)=u$. □

One way to see that this representation is not spin is to look at the *binary tetrahedral group $2T$*, defined to be the preimage of $A_4\subset\mathrm{SO}_3$ under the double cover $\mathrm{Spin}_3\twoheadrightarrow\mathrm{SO}_3$. If V_λ were spin, $2T$ would be a split extension of A_4 by μ_2, but it is not split.

4.5.1.1. *Class D, spinless case.* If A_4 does not mix with the symmetry type, our ansatz reduces to that of Freed-Hopkins, which reduces the computation of these A_4-equivariant phase homology groups to the computation of $[MTSpin\wedge(BA_4)^{3-V_\lambda},\Sigma^5 I_\mathbb{Z}]$.

Theorem 4.5.4. *The first few spin bordism groups of $X := (BA_4)^{3-V_\lambda}$ are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}}(X)&\cong\mathbb{Z}\\
\widetilde\Omega_1^{\mathrm{Spin}}(X)&\cong\mathbb{Z}/3\\
\widetilde\Omega_2^{\mathrm{Spin}}(X)&\cong 0\\
\widetilde\Omega_3^{\mathrm{Spin}}(X)&\cong\mathbb{Z}/6\\
\widetilde\Omega_4^{\mathrm{Spin}}(X)&\cong\mathbb{Z}\\
\widetilde\Omega_5^{\mathrm{Spin}}(X)&\cong\mathbb{Z}/18\oplus\mathbb{Z}/2\\
\widetilde\Omega_6^{\mathrm{Spin}}(X)&\cong\mathbb{Z}/2\\
\widetilde\Omega_7^{\mathrm{Spin}}(X)&\cong\mathbb{Z}/9.
\end{aligned}$$

Thus if f_0^D denotes the A_4-equivariant local system of symmetry types for this case, $Ph_0^{A_4}(\mathbb{R}^3, f_0^D) = 0$.

Proof. At the prime 2, we use the Adams spectral sequence; if p is an odd prime, the map $\widetilde{\Omega}_*^{\mathrm{Spin}}(X) \to \widetilde{\Omega}_*^{\mathrm{SO}}(X)$ is an isomorphism on p-torsion, and we will determine the p-torsion part of $\widetilde{\Omega}_*^{\mathrm{SO}}(X)$.

First, the 2-primary piece. Letting U denote the mod 2 Thom class as usual, $\mathrm{Sq}^1(U) = 0$ and $\mathrm{Sq}^2(U) = Uu$. This and Proposition 4.5.1 allow us to determine the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X;\mathbb{Z}/2)$ in low degrees, as depicted in Figure 19, left.

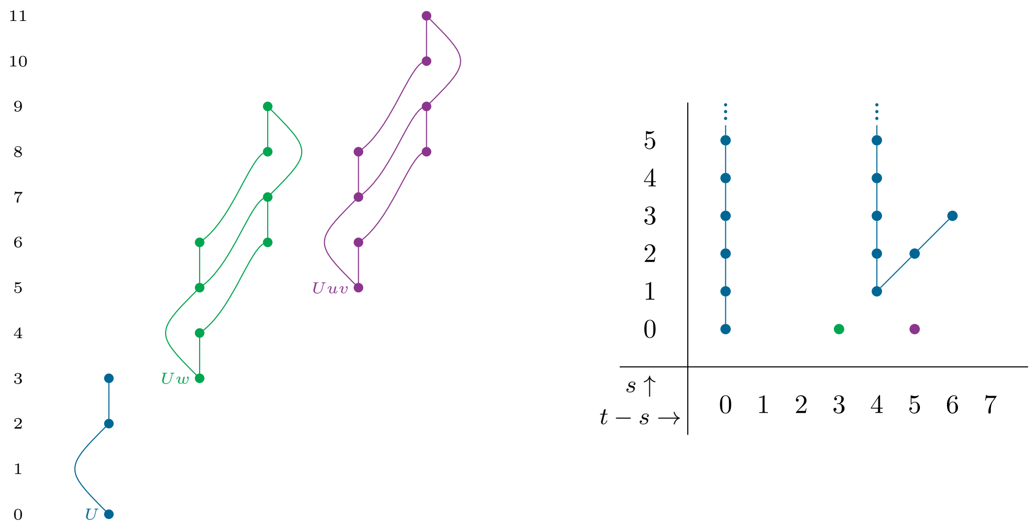


Figure 19. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BA_4)^{3-V_\lambda};\mathbb{Z}/2)$ in low degrees. This submodule contains all elements of degree at most 8. Right: the E_2-page of the Adams spectral sequence calculating $\widetilde{ko}_*((BA_4)^{3-V_\lambda})$, given by $\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(\widetilde{H}^*((BA_4)^{3-V_\lambda};\mathbb{Z}/2),\mathbb{Z}/2)$.

Hence as $\mathcal{A}(1)$-modules,

$$\widetilde{H}^*(X;\mathbb{Z}/2) \cong \mathring{\mathcal{O}} \oplus \Sigma^3\mathcal{A}(1) \oplus \Sigma^5\mathcal{A}(1) \oplus P, \tag{4.5.5}$$

where P is 8-connected. Because we only care about degrees 6 and below, P is irrelevant for us, and for the remaining summands in (4.5.5), $\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(-,\mathbb{Z}/2)$ has already been computed. For $\Sigma^k\mathcal{A}(1)$, there's a single $\mathbb{Z}/2$ with $s = 0$, $t = k$; for $\mathring{\mathcal{O}}$, see [**BC18**, Figure 29]. We put this together and display the E_2-page for our spectral sequence in Figure 19, right. A combination of h_0-equivariance and Margolis' theorem (Theorem 4.3.14) rules out nontrivial differentials and hidden extensions. Therefore the 2-primary part of $\widetilde{\Omega}_k^{\mathrm{Spin}}(X)$ has a single free summand each in degrees 0 and 4, is 0 in degrees 1 and 2, is $\mathbb{Z}/2$ in degrees 3 and 6, and is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ in degree 5.

For the odd-primary part, we use the fact that $\Omega_*^{\mathrm{Spin}} \to \Omega_*^{\mathrm{SO}}$ is an equivalence after inverting 2. Moreover, because λ factors through SO_3, $V_\lambda \to BA_4$ is orientable, so there is a Thom isomorphism

$\widetilde{\Omega}_*^{\mathrm{SO}}(X) \xrightarrow{\cong} \Omega_*^{\mathrm{SO}}(BA_4)$. Hence we just need the odd-primary part of $\Omega_*^{\mathrm{SO}}(BA_4)$, which is isomorphic to the odd-primary part of $\Omega_*^{\mathrm{Spin}}(BA_4)$. In the degrees we care about, this is isomorphic to $ko_*(BA_4)$, and Bruner-Greenlees [**BG10**, §7.7.E] show that the odd-primary torsion in $ko_*(BA_4)$ below degree 6 consists of $\mathbb{Z}/3$ summands in degrees 1 and 3 and $\mathbb{Z}/9$ summands in degrees 5 and 7. ☐

4.5.1.2. *Class D, spin-*$1/2$ *case.* In this case, the symmetries mix as specified by the group extension giving the binary tetrahedral group.

Theorem 4.5.6. *The A_4-equivariant phase homology group for the class D, spin-$1/2$ symmetry type in 3d is trivial.*

Proof. Let $f_{1/2}^D$ denote the local system on $\mathbb{R}^3$ assigned to this symmetry type. Since V_λ is not pin$^-$ (if it were, it would be pin$^-$ and orientable, hence spin), Theorem 4.2.11 says $Ph_0^{A_4}(\mathbb{R}^3; f_{1/2}^D) \cong [\mathit{MTSpin} \wedge (BA_4)_+, \Sigma^5 I_{\mathbb{Z}}]$. Bruner-Greenlees [**BG10**, §7.7.E] show $ko_4(BA_4) \cong \mathbb{Z}$ and $ko_5(BA_4)$ is torsion, so this phase homology group vanishes. ☐

4.5.1.3. *Class A.* Let f_0^A and $f_{1/2}^A$ be the A_4-equivariant local systems of symmetry types for spinless, resp. spin-$1/2$ fermions in class A.

Lemma 4.5.7. *$V_\lambda \to BA_4$ is not pin^c.*

Proof. If $\beta\colon H^2(-;\mathbb{Z}/2) \to H^3(-;\mathbb{Z})$ denotes the integral Bockstein, we want to show $\beta w_2(V_\lambda) \neq 0$. By Lemma 4.3.20, it suffices to show $\mathrm{Sq}^1(w_2(V_\lambda)) \neq 0$. Lemma 4.5.3 gives $w_2(V_\lambda) = b$, and $\mathrm{Sq}^1 b = ab + c$. ☐

Therefore for spin-$1/2$ fermions, Theorem 4.2.24 computes $Ph_*^{A_4}(\mathbb{R}^3; f_{1/2}^A)$ in terms of the spin^c bordism of $(BA_4)^{\mathrm{Det}(V_\lambda)-1}$. Since V_λ is orientable, this is isomorphic to the spin^c bordism of BA_4. For spinless fermions, we use $(BA_4)^{3-V_\lambda}$, as usual.

Theorem 4.5.8. *The low-degree spin^c bordism groups of $X := (BA_4)^{3-V_\lambda}$ and BA_4 are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z} & \Omega_0^{\mathrm{Spin}^c}(BA_4) &\cong \mathbb{Z}\\
\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/3 & \Omega_1^{\mathrm{Spin}^c}(BA_4) &\cong \mathbb{Z}/3\\
\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z} & \Omega_2^{\mathrm{Spin}^c}(BA_4) &\cong \mathbb{Z}\oplus\mathbb{Z}/2\\
\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/6\oplus\mathbb{Z}/3 & \Omega_3^{\mathrm{Spin}^c}(BA_4) &\cong \mathbb{Z}/6\oplus\mathbb{Z}/3\\
\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}^2 & \Omega_4^{\mathrm{Spin}^c}(BA_4) &\cong \mathbb{Z}^2,
\end{aligned}$$

and in both cases, $\Omega_5^{\mathrm{Spin}^c}$ is torsion. Hence both $Ph_0^{A_4}(\mathbb{R}^3; f_0^A)$ and $Ph_0^{A_4}(\mathbb{R}^3; f_{1/2}^A)$ vanish.

Proof. We use the equivalence $MTSpin^c \simeq ku \vee \Sigma^4 ku$ in degrees below 8, then the Adams spectral sequence over $\mathcal{E}(1)$ to compute ku-homology at the prime 2.

For the case of spinless fermions, use the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X;\mathbb{Z}/2)$ from (4.5.5) (drawn in Figure 19, left) to compute that the $\mathcal{E}(1)$-module structure is

$$\widetilde{H}^*(X;\mathbb{Z}/2) \cong \eth \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^5\mathcal{E}(1) \oplus \Sigma^5\mathcal{E}(1) \oplus P, \tag{4.5.9}$$

where P is 6-connected. We draw this in Figure 20, left. We computed $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(\eth, \mathbb{Z}/2)$ in (4.4.56), and P is too high-degree to be relevant to us, so the E_2-page of the Adams spectral sequence for $\widetilde{ku}_*(X)$ is given in Figure 20, right. Margolis' theorem (Theorem 4.3.14) implies this spectral sequence collapses and there are no extension problems, so we conclude.

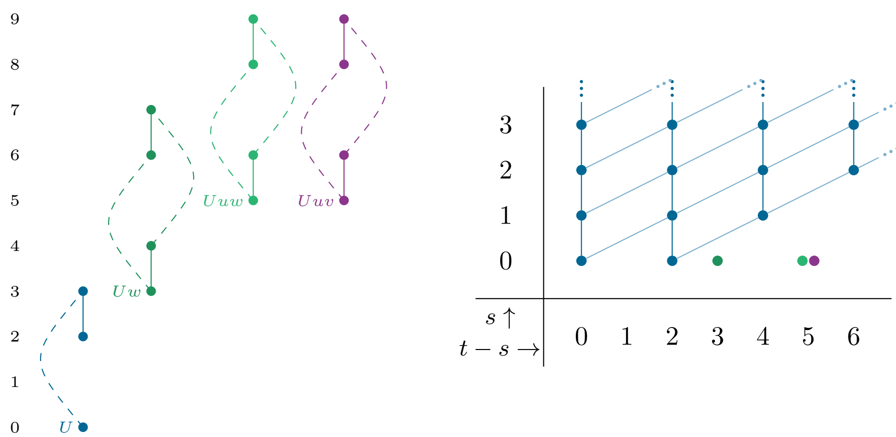


Figure 20. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*((BA_4)^{3-V_\lambda};\mathbb{Z}/2)$ in low degrees. The picture includes all elements in degrees 6 and below. Right: $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(\widetilde{H}^*((BA_4)^{3-V_\lambda};\mathbb{Z}/2),\mathbb{Z}/2)$, the E_2-page of the Adams spectral sequence for $\widetilde{ku}_*((BA_4)^{3-V_\lambda})$.

On to the spin-1/2 case. As before, $ku_*(BA_4)$ splits as $ku_*(\mathrm{pt}) \oplus \widetilde{ku}_*(BA_4)$, and we focus on the latter. Bruner-Greenlees [**BG03**, §2.6] show that 2-locally, there is an equivalence

$$ku \wedge BA_4 \simeq (ku \wedge \Sigma^2 BC_2) \vee \bigvee_\alpha \Sigma^{n_\alpha} H\mathbb{Z}/2 \tag{4.5.10}$$

for some collection of integers α; moreover, their calculation of $ku^*(BA_4)$ [**BG03**, Theorem 2.6.3] implies the only $n_\alpha < 8$ (i.e. the ones relevant for us) are $n_1 = 2$ and $n_2 = 6$. This, together with Hashimoto's computation of $\widetilde{ku}_*(B\mathbb{Z}/2)$ [**Has83**, Theorem 3.1], tells us $ku_*(BA_4)^\wedge_2$ in the degrees we need.

We still need to determine the odd-primary torsion.

Lemma 4.5.11. *Let p be an odd prime; then, the inclusion $\mathbb{Z}/3 \hookrightarrow A_4$ sending a generator to* $(1\ 2\ 3)$ *induces a p-primary stable equivalence $\Sigma^\infty(B\mathbb{Z}/3)_+ \to \Sigma^\infty(BA_4)_+$.*

Proof. Since $|A_4| = 2^2 \cdot 3$, for any $p \geq 5$, the maps $BA_4 \to \mathrm{pt}$ and $B\mathbb{Z}/3 \to \mathrm{pt}$ are p-local stable equivalences, so we only have to address $p = 3$. In this case, Lemma 4.3.19 implies the inclusion $j\colon \mathbb{Z}/3 \hookrightarrow A_4$ as the subgroup generated by $(1\ 2\ 3)$ induces an isomorphism $H^*(BA_4;\mathbb{Z}/3) \to H^*(B\mathbb{Z}/3;\mathbb{Z}/3)$, so we conclude by the mod p Whitehead theorem [**Ser53**, Chapitre III, Théorème 3]. $\square$

The Thom isomorphism theorem then implies $\widetilde{H}^*(X;\mathbb{Z}[1/2]) \to \widetilde{H}^*((B\mathbb{Z}/3)^{3-j^*V_\lambda};\mathbb{Z}[1/2])$ is an isomorphism, so arguing in a similar way, there is a p-primary stable equivalence $X_\lambda \simeq (B\mathbb{Z}/3)^{3-V_\lambda}$. Thus, for the purpose of computing the odd-torsion subgroups of $\Omega_*^{\mathrm{Spin}^c}(BA_4)$ and $\widetilde{\Omega}_*^{\mathrm{Spin}^c}(X)$, we can just work with $\mathbb{Z}/3$.

As a $\mathbb{Z}/3$-representation, j^*V_λ is isomorphic to the direct sum of a trivial representation and the real 2-dimensional representation given by rotation. Each of these is spin^c, the latter because it is unitary, so there is a Thom isomorphism $\mathit{MTSpin}^c \wedge (B\mathbb{Z}/3)^{3-j^*V_\lambda} \cong \mathit{MTSpin}^c \wedge (B\mathbb{Z}/3)_+$, so in both the spinless and spin-1/2 cases, we just need the 3-torsion in $\Omega_*^{\mathrm{Spin}^c}(B\mathbb{Z}/3)$, which we computed in Theorem 4.4.15. $\square$

4.5.2. Pyritohedral symmetry. *Pyritohedral symmetry* is the action of $G \coloneqq A_4 \times \mathbb{Z}/2$ on $\mathbb{R}^3$ in which A_4 acts as the orientation-preserving symmetries of a tetrahedron and $\mathbb{Z}/2$ acts through inversion; let λ denote this representation and $V_\lambda \to BG$ be the associated vector bundle. Because G splits as a direct product, it is easier to analyze than full tetrahedral symmetry (i.e. chiral tetrahedral symmetry together with a reflection), as we will see in this and the next section.

4.5.2.1. *Spinless case.* Let $X \coloneqq (BG)^{3-V_\lambda}$. By the twisted Künneth formula, $H^*(X)$ is 2-torsion; therefore $\widetilde{\Omega}_*^{\mathrm{Spin}}(X)$ also lacks odd-primary torsion. so we just have to work with the Adams spectral sequence at $p = 2$. In the rest of this section, all cohomology is with $\mathbb{Z}/2$ coefficients unless otherwise stated.

Proposition 4.5.12. *The first several spin bordism groups of $(BG)^{3-V_\lambda}$ are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong \mathbb Z/2\\
\widetilde\Omega_1^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong 0\\
\widetilde\Omega_2^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong \mathbb Z/2\\
\widetilde\Omega_3^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong \mathbb Z/2\\
\widetilde\Omega_4^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong (\mathbb Z/2)^{\oplus 3}\\
\widetilde\Omega_5^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong (\mathbb Z/2)^{\oplus 3}\\
\widetilde\Omega_6^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong \mathbb Z/16\oplus(\mathbb Z/2)^{\oplus 2}\\
\widetilde\Omega_7^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong (\mathbb Z/2)^{\oplus 2}.
\end{aligned}$$

Proof. We employ a trick to reduce the amount of direct computations. We will replace $(3-V_\lambda)\to BG$ with a virtual vector bundle $E\to BG$ with the same first two Stiefel-Whitney classes, but which splits as an exterior sum over BA_4 and $B\mathbb Z/2$. The Thom spectrum $(BG)^E$ has two nice properties: the Adams E_2-page for calculating $\widetilde{ko}_*((BG)^E)$ is isomorphic to that of $\widetilde{ko}_*(X)$, but $(BG)^E$ also splits as a smash product of Thom spectra over BA_4 and $B\mathbb Z/2$, simplifying the calculation of said E_2-page. Because we do not construct a map from $\widetilde{ko}_*((BG)^E)$ to $\widetilde{ko}_*((BG)^{3-V_\lambda})$ or vice versa, this isomorphism does not allow us to deduce any differentials, but we will see that all differentials in range vanish for formal reasons, so this is no problem.

The Künneth formula and Proposition 4.5.1 together imply

$$H^*(BG)\cong\mathbb Z/2[x,u,v,w]/(u^3+v^2+w^2+vw), \tag{4.5.13}$$

where $|x|=1$, $|u|=2$, and $|v|=|w|=3$, and that $\mathrm{Sq}(x)=x+x^2$ and the Steenrod squares of u, v, and w are as in Proposition 4.5.1.

Lemma 4.5.14. *The first two Steifel-Whitney classes of V are $w_1(V_\lambda)=x$ and $w_2(V_\lambda)=u+x^2$.*

Proof. Since this representation contains orientation-reversing symmetries, $w_1(V_\lambda)$ must be nonzero, so is x. For w_2, we saw in Lemma 4.5.3 that when one restricts to $A_4\subset A_4\times\mathbb Z/2$, one has $w_2(V_\lambda|_{BA_4})=u$; when one restricts to $\mathbb Z/2$, this is 3 copies of the sign representation, hence has $w_2(V_\lambda|_{B\mathbb Z/2})=x^2$. $\square$

Let $E\to BG$ be the virtual vector bundle

$$E \coloneqq 4-(V_\lambda|_{BA_4}\boxplus -\sigma), \tag{4.5.15}$$

where $\sigma \to B\mathbb{Z}/2$ is the tautological line bundle. The Whitney sum formula implies for $i = 1, 2$, $w_i(E) = w_i(3 - V_\lambda)$. Feeding this to the Thom isomorphism gives isomorphisms of $\mathcal{A}(1)$-modules

$$\widetilde{H}^*((BG)^{3-V_\lambda}) \cong \widetilde{H}^*((BG)^E) \tag{4.5.16}$$

hence also isomorphisms of the E_2-pages of the corresponding Adams spectral sequences. Because $E \to BG$ is an external sum,

$$(BG)^E \simeq (BA_4)^{3-V_\lambda} \wedge (B\mathbb{Z}/2)^{\sigma-1}. \tag{4.5.17}$$

We know the $\mathcal{A}(1)$-module structures on the low-degree cohomology of both summands, and the Künneth formula tells us to tensor them together (over $\mathbb{Z}/2$) to determine the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BG)^E)$.

In (4.5.5), we computed the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BA_4)^{3-V_\lambda})$ in low degrees, and split off two $\Sigma^k\mathcal{A}(1)$ summands. Margolis' theorem (Theorem 4.3.14) promotes that to a splitting of spectra

$$ko \wedge (BA_4)^{3-V_\lambda} \simeq \Sigma^3 H\mathbb{Z}/2 \vee \Sigma^5 H\mathbb{Z}/2 \vee Y, \tag{4.5.18}$$

such that as an $\mathcal{A}$-module,

$$\widetilde{H}^*(Y) \cong \mathcal{A} \otimes_{\mathcal{A}(1)} (\hat{\mathcal{O}} \oplus P), \tag{4.5.19}$$

where P is 7-connected. When we smash $(B\mathbb{Z}/2)^{\sigma-1}$ back in, each $\Sigma^k H\mathbb{Z}/2 \wedge (B\mathbb{Z}/2)^{\sigma-1}$ contributes a summand of $\widetilde{H}_{n-k}((B\mathbb{Z}/2)^{\sigma-1})$ to $\widetilde{ko}_n((BG)^E)$, i.e. a $\mathbb{Z}/2$-summand in each degree $\ell \geq k$. The upshot for $\mathcal{A}(1)$-modules is

$$\Sigma^k\mathcal{A}(1) \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}) \cong \bigoplus_{\ell \geq k} \Sigma^\ell H\mathbb{Z}/2. \tag{4.5.20}$$

By (4.5.16), these summands are also present in $\widetilde{H}^*((BG)^{3-V_\lambda})$, and Margolis' theorem lifts this to split off corresponding $\Sigma^\ell H\mathbb{Z}/2$ summands. Therefore there is a spectrum Y' such that

$$\widetilde{ko}_n((BG)^E) \cong \pi_n(Y') \oplus \widetilde{H}_{n-3}((B\mathbb{Z}/2)^{\sigma-1}) \oplus \widetilde{H}_{n-5}((B\mathbb{Z}/2)^{\sigma-1}) \tag{4.5.21}$$

and as $\mathcal{A}$-modules,

$$\widetilde{H}^*(Y') \cong \mathcal{A} \otimes_{\mathcal{A}(1)} (\hat{\mathcal{O}} \oplus P) \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}). \tag{4.5.22}$$

The change-of-rings theorem (1.1.43) thus applies to the E_2-page of the Adams spectral sequence calculating $\pi_*(Y')$, yielding

$$E_2^{s,t} \cong \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}((¿ \oplus P) \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}), \mathbb{Z}/2). \tag{4.5.23}$$

We will work with this spectral sequence, adding in the summands corresponding to $\Sigma^3 H\mathbb{Z}/2$ and $\Sigma^5 H\mathbb{Z}/2$ afterwards.

Our first order of business is to compute the tensor product in (4.5.23). The $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1})$ can be found in [**BC18**, Figure 4].

Lemma 4.5.24. *There is an isomorphism of $\mathcal{A}(1)$-modules $¿ \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}) \cong \mathcal{A}(1) \oplus \Sigma^2 R_0 \oplus \Sigma^4\mathcal{A}(1) \oplus P$, where P is 7-connected.*

Proof. Compute directly, by hand or by computer. ☐

By (4.5.21) and (4.5.22), we can work with (4.5.23), then add in the $\mathbb{Z}/2$ summands coming from the $\Sigma^k H\mathbb{Z}/2$ summands at the end. Lemma 4.5.24 tells us the E_2-page of (4.5.23) is

$$E_2^{s,t} \cong \mathrm{Ext}(\mathcal{A}(1) \oplus \Sigma^2 R_0 \oplus \Sigma^4\mathcal{A}(1) \oplus P). \tag{4.5.25}$$

Since P is 7-connected, its Ext is concentrated in degrees irrelevant to us, and we ignore it. $\mathrm{Ext}(\Sigma^2 R_0)$ is computed in the degrees we need by Beaudry-Campbell [**BC18**, Figures 23, 24]; using this, we draw the E_2-page of (4.5.23) in Figure 21. Margolis' theorem and h_1-equivariance of differentials immediately imply

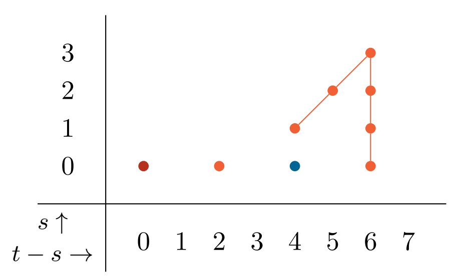


Figure 21. The Adams spectral sequence (4.5.25) computing $\pi_*(Y')$.

there are no nontrivial differentials or extension problems below degree 8, so we conclude. ☐

4.5.2.2. *Class D, spin-1/2 case.* Let $f^D_{1/2}$ denote the equivariant local system of symmetry types corresponding to spin-1/2 fermions for a pyritohedral symmetry in class D. Theorem 4.2.11 computes the equivariant phase homology associated to $f^D_{1/2}$ in terms of the spin bordism of $X := (BA_4 \times B\mathbb{Z}/2)^{\mathrm{Det}(V_\lambda)-1}$. The isomorphism $\mathrm{Det}(V_\lambda) \cong 0 \boxplus \sigma$ provides an isomorphism $X \simeq (BA_4)_+ \wedge (B\mathbb{Z}/2)^{\sigma-1}$, Lemma 4.3.22 thus implies the spin bordism of this spectrum computes the pin$^-$ bordism of BA_4, which could be independently interesting.

Theorem 4.5.26. *The first few spin bordism groups of X are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}}(X) &\cong \mathbb Z/2\\
\widetilde\Omega_1^{\mathrm{Spin}}(X) &\cong \mathbb Z/2\\
\widetilde\Omega_2^{\mathrm{Spin}}(X) &\cong \mathbb Z/8\oplus\mathbb Z/2\\
\widetilde\Omega_3^{\mathrm{Spin}}(X) &\cong \mathbb Z/2\oplus\mathbb Z/2\\
\widetilde\Omega_4^{\mathrm{Spin}}(X) &\cong \mathbb Z/2.
\end{aligned}$$

Since $\widetilde\Omega_5^{\mathrm{Spin}}(X)$ is torsion by Lemma 4.3.16, $Ph_0^{A_4\times\mathbb Z/2}(\mathbb R^3; f_{1/2}^D)\cong\mathbb Z/2$.

Proof. By the twisted Künneth formula, $\widetilde H^*(X)$ has no odd-primary torsion, and therefore neither does $\widetilde\Omega_*^{\mathrm{Spin}}(X)$, so it suffices to work at the prime 2, which we do.

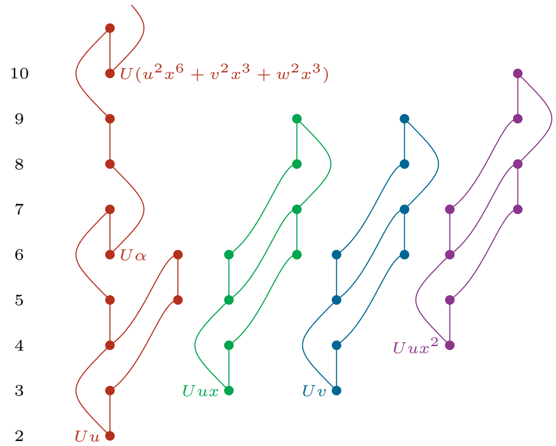
Use Lemma 4.3.22 to split $X\simeq(B\mathbb Z/2)^{\sigma-1}\vee M$, where the map $\widetilde H^*(M;\mathbb Z/2)\to\widetilde H^*(X;\mathbb Z/2)$ is injective with image a complimentary subspace to $\mathbb Z/2\cdot\{Ux^k\mid k\ge 0\}$.

As usual, $w_1(\mathrm{Det}(V_\lambda)-1)=w_1(V_\lambda)=x$ and $w_2(\mathrm{Det}(V_\lambda)-1)=0$. We also need to know the $\mathcal A$-action on $H^*(BG;\mathbb Z/2)$; the Künneth formula determines this using as input the $\mathcal A$-action on $H^*(BA_4;\mathbb Z/2)$, which we computed in Proposition 4.5.1, and the $\mathcal A$-action on $H^*(B\mathbb Z/2;\mathbb Z/2)$, which is standard. Using this, we can determine the $\mathcal A(1)$-module structure on $\widetilde H^*(M;\mathbb Z/2)$. We obtain an isomorphism of $\mathcal A(1)$-modules

$$\widetilde H^*(M;\mathbb Z/2)\cong\Sigma^2R_3\oplus\Sigma^3\mathcal A(1)\oplus\Sigma^3\mathcal A(1)\oplus\Sigma^4\mathcal A(1)\oplus P, \tag{4.5.27}$$

where P is 4-connected. We will see in Figure 22, right, that for $t-s\le 4$, $E_2^{s,t}$ is concentrated in Adams filtration 0; this and the 4-connectedness of P imply its contribution to the E_2-page cannot affect the spectral sequence in degrees $t-s\le 4$, which is all we need. We draw these summands, except for P, in Figure 22, left.

Freed-Hopkins [**FH16a**, Figure 5, case $s=3$] and Beaudry-Campbell [**BC18**, Figures 32, 33] calculate $\mathrm{Ext}(R_3)$ in the range we need, and we can draw the E_2-page of the Adams spectral sequence in Figure 22, right. This collapses, so we add in the pin$^-$ bordism summands we need from [**ABP69, KT90b**] to obtain the groups in the theorem. $\square$



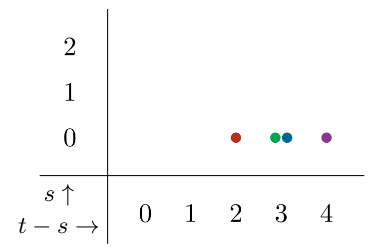


FIGURE 22. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ in low degrees. This picture includes all summands in degrees 4 and below. Here $\alpha \coloneqq u^2x^2+v^2+w^2$. Right: the E_2-page of the corresponding Adams spectral sequence.

4.5.2.3. *Class A, spinless case.* Let f_0^A denote the equivariant local system of symmetry types corresponding to spinless fermions in class A and $X \coloneqq (BA_4 \times B\mathbb{Z}/2)^{3-V_\lambda}$; then we saw that $Ph_0^{A_4\times\mathbb{Z}/2}(\mathbb{R}^3; f_0^A)$ is determined by $\widetilde{\Omega}_*^{\mathrm{Spin}^c}(X)$.

Theorem 4.5.28. *The first few spin^c bordism groups of X are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X) &\cong 0\\
\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/4\oplus(\mathbb{Z}/2)^{\oplus 3}\\
\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X) &\cong (\mathbb{Z}/2)^{\oplus 3}\\
\widetilde{\Omega}_6^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/8\oplus(\mathbb{Z}/2)^{\oplus 6}\\
\widetilde{\Omega}_7^{\mathrm{Spin}^c}(X) &\cong (\mathbb{Z}/2)^{\oplus 5},
\end{aligned}$$

so $Ph_0^{A_4\times\mathbb{Z}/2}(\mathbb{R}^3; f_0^A) \cong \mathbb{Z}/4\oplus(\mathbb{Z}/2)^{\oplus 3}$.

PROOF. The twisted Thom isomorphism and twisted Künneth formula imply $\widetilde{H}^*(X;\mathbb{Z})$ is 2-torsion. Therefore for any odd prime p, the mod p Whitehead theorem [Ser53, Chapitre III, Théorème 3] implies $\widetilde{\Omega}_*^{\mathrm{Spin}^c}(X)$ also has no p-torsion. This leaves only $p=2$, for which we use the Adams spectral sequence over $\mathcal{E}(1)$.

We determined the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BG)^{3-V_\lambda})$ as given in (4.5.25), together with an $\Sigma^\ell\mathcal{A}(1)$ for $\ell = 3, 4$, and two $\Sigma^\ell\mathcal{A}(1)$ summands for $\ell \geq 5$. This determines the $\mathcal{E}(1)$-module structure: as $\mathcal{E}(1)$-modules, $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$, and $R_0 \cong H$, so

$$\widetilde{H}^*(X;\mathbb{Z}/2) \cong \Sigma^2 H \oplus V' \otimes_{\mathbb{Z}/2} \mathcal{E}(1) \oplus P, \tag{4.5.29}$$

where V' is a graded $\mathbb{Z}/2$-vector space with a basis of homogeneous elements in degrees 0, 2, 3, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, and 7, and P is 7-connected. Therefore the E_2-page is as drawn in Figure 23. By Margolis'

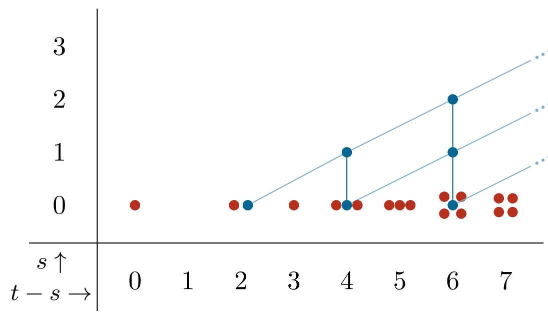


FIGURE 23. The Adams spectral sequence computing $\widetilde{ku}_*((BG)^{3-V_\lambda})$.

theorem, there are no nontrivial differentials or extension problems in this range. □

4.5.2.4. *Class A, spin-1/2 case.* To compute the $(A_4 \times \mathbb{Z}/2)$-equivariant phase homology groups for the local system $f^A_{1/2}$ specified by the spin-1/2 extension in class A Theorem 4.2.24 asks us to investigate the spin^c bordism of $X \coloneqq (BA_4 \times B\mathbb{Z}/2)^{\mathrm{Det}(V_\lambda)-1} \simeq (BA_4 \times B\mathbb{Z}/2)^{0 \boxplus \sigma - 1}$; we know V_λ is not pin^c because we saw in Lemma 4.5.7 that the pullback of V_λ along $BA_4 \to BA_4 \times B\mathbb{Z}/2$ is not pin^c.

Theorem 4.5.30. *The first few spin^c bordism groups of X are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X) &\cong 0\\
\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/4 \oplus \mathbb{Z}/2\\
\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X) &\cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 3}
\end{aligned}$$

By Lemma 4.3.16, $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X)$ is torsion, so $Ph_0^{A_4\times\mathbb{Z}/2}(\mathbb{R}^3; f^A_{1/2}) \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 3}$.

Proof. We reuse our work from §4.5.2.2. We saw that $X \simeq (B\mathbb{Z}/2)^{\sigma-1} \vee M$, and we gave the low-degree cohomology of M as an $\mathcal{A}(1)$-module in (4.5.27) (and drew it in Figure 22, left). This determines the $\mathcal{E}(1)$-module structure on it, so we can calculate spin^c bordism of M using the Adams spectral sequence. For the other summand, we have $MTSpin^c \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq MTPin^c$, so we direct-sum in the pin^c bordism groups computed by Bahri-Gilkey [**BG87a, BG87b**].

There are isomorphisms of $\mathcal{E}(1)$-modules $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$ and $R_3 \cong \Sigma^2\mathcal{E}(1) \oplus \Sigma^4 R_0$. Therefore as an $\mathcal{E}(1)$-module,

$$\widetilde{H}^*(M;\mathbb{Z}/2) \cong \Sigma^2\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^4 R_0 \oplus \Sigma^4\mathcal{E}(1) \oplus P, \tag{4.5.31}$$

where P is 4-connected. As usual for these cases, we will see that $\mathrm{Ext}(\widetilde{H}^*(M;\mathbb{Z}/2),\mathbb{Z}/2)$ has no nonzero elements with $t-s=4$ and $s>1$, so P does not affect our calculations. See Figure 24, left, for a picture of the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$. Look up $\mathrm{Ext}(\Sigma^4 R_0)$ in Proposition 4.4.49 to obtain the E_2-page of

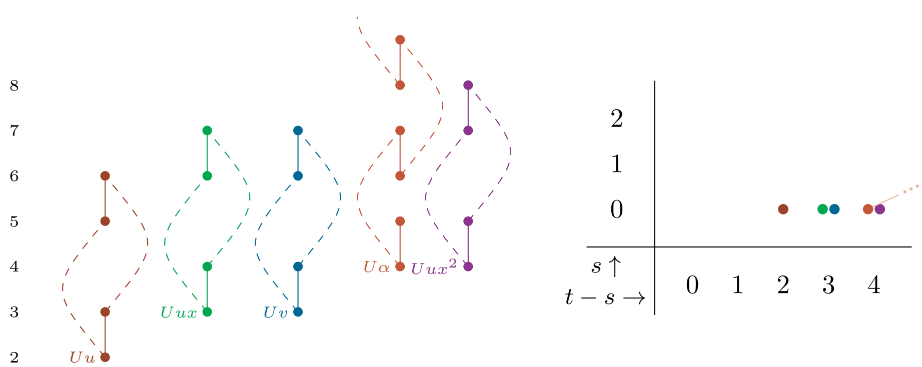


Figure 24. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ in low degrees. This picture includes all summands in degrees 4 and below. Here $\alpha \coloneqq u^2+vx+wx$. Right: the E_2-page of the corresponding Adams spectral sequence.

the Adams spectral sequence as in Figure 24, right. This collapses, so we add in the pin^c bordism summands and conclude. $\square$

I could get used to Adams spectral sequences like this one. But alas, they are not all this easy, as we will see in the next section.

4.5.3. Full tetrahedral symmetry. The full group of symmetries of the tetrahedron, including reflections, is the symmetric group S_4, acting via the representation $\lambda\colon S_4 \to \mathrm{O}_3$, which is isomorphic to the quotient of the four-dimensional real permutation representation by the fixed line $\mathbb{R}\cdot(1,1,1,1)$.

Proposition 4.5.32 ([**Ngu09**, §2.3])**.** *$H^*(BS_4;\mathbb{Z}/2) \cong \mathbb{Z}/2[a,b,c]/(ac)$, with $|a| = 1$, $|b| = 2$, and $|c| = 3$. The Steenrod squares of the generators are $\mathrm{Sq}(a) = a + a^2$, $\mathrm{Sq}(b) = b + ab + c + b^2$, and $\mathrm{Sq}(c) = c + bc + c^2$.*[26]

Let $V_\lambda \to BS_4$ denote the associated vector bundle to λ.

Proposition 4.5.33. *$w_1(V_\lambda) = a$, $w_2(V_\lambda) = b$, and $w_3(V_\lambda) = c$.*

Proof. Since λ does not factor through $\mathrm{SO}_3 \subset \mathrm{O}_3$, V_λ is unorientable. Thus $w_1(V_\lambda) \neq 0$, and a is the only nonzero element of $H^1(BS_4;\mathbb{Z}/2)$, so $w_1(V_\lambda) = a$. For w_2, we calculated in Lemma 4.5.3 that $w_2(V_\lambda|_{A_4}) \neq 0$, so w_2 cannot vanish in BS_4. Our options are a^2, b, and $a^2 + b$. Let $\mathbb{Z}/2 \subset S_4$ be generated by a transposition; then as a $\mathbb{Z}/2$-representation $\lambda \cong \mathbb{R}^2 \oplus \sigma$, so $w_2(V_\lambda|_{\mathbb{Z}/2}) = 0$. The map $H^*(BS_4;\mathbb{Z}/2) \to H^*(B\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]$ sends $b, c \mapsto 0$ and $a \mapsto x$, so the constraint $w_2(V_\lambda|_{\mathbb{Z}/2}) = 0$ rules out $w_2(V_\lambda) = a^2$ and $w_2(V_\lambda) = a^2 + b$, forcing us to conclude $w_2(V_\lambda) = b$. Finally, $w_3(V_\lambda) = c$ follows from the Wu formula. $\square$

We need the next calculation to determine the odd-primary torsion subgroups of the phase homology groups that we calculate.

Lemma 4.5.34. *Suppose $V \to BS_4$ is a rank-zero virtual vector bundle with $w_1(V) = x$. Then the inclusion $i\colon S_3 \hookrightarrow S_4$ defines an isomorphism*

$$\widetilde{H}_*((BS_3)^{i^*V}) \otimes \mathbb{Z}[1/2] \to \widetilde{H}_*((BS_4)^V) \otimes \mathbb{Z}[1/2]. \tag{4.5.35}$$

Proof. The commutative diagram of short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & A_3 \arrow[r] \arrow[d, hook] & S_3 \arrow[r] \arrow[d, hook, "i"] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & A_4 \arrow[r] & S_4 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 1 \end{tikzcd} \tag{4.5.36}$$

induces a map between their Lyndon-Hochschild-Serre spectral sequences with signatures

$$E^2_{p,q} = H_p(B\mathbb{Z}/2; \underline{H_q(BA_k;\mathbb{Z}[1/2])} \otimes (\mathbb{Z}[1/2])_x) \Longrightarrow H_{p+q}(BS_k; (\mathbb{Z}[1/2])_{w_1(V)}), \tag{4.5.37}$$

where $\underline{H_q(BA_4;\mathbb{Z}[1/2])}$ means the local system on $B\mathbb{Z}/2$ induced by the action of $\mathbb{Z}/2$ on A_k as specified by the extension $1 \to A_k \to S_k \to \mathbb{Z}/2 \to 1$, and x is the generator of $H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$.

We claim the map on these spectral sequences is an isomorphism on E^2-pages. By Lemma 4.5.11, the map $H_*(BA_4;\mathbb{Z}[1/2]) \to H_*(BA_3;\mathbb{Z}[1/2])$ is an isomorphism, and this isomorphism intertwines the $\mathbb{Z}/2$-actions

[26] The ring structure on $H^*(BS_4;\mathbb{Z}/2)$ was known earlier, due to Cardenas [**Car65**]; see [**AM04**, Example VI.1.13].

on $\underline{H_*(BA_k;\mathbb{Z}[1/2])}\otimes(\mathbb{Z}[1/2])_x$, because (4.5.36) commutes. Therefore it induces an isomorphism on all E^r-pages, hence also on what these spectral sequences converge to. □

The top row in (4.5.36) can be identified with $1\to\mathbb{Z}/3\to D_6\to\mathbb{Z}/2\to 1$, so by the same lines of reasoning as in Propositions 4.4.24 and 4.4.25 we deduce

(4.5.38a)
$$\widetilde{\Omega}_k^{\mathrm{Spin}}((BS_4)^V)\otimes\mathbb{Z}[1/2]\cong\begin{cases}\mathbb{Z}/3, & k=1\\ 0, & k=0,2,3,4\end{cases}$$

(4.5.38b)
$$\widetilde{\Omega}_k^{\mathrm{Spin}^c}((BS_4)^V)\otimes\mathbb{Z}[1/2]\cong\begin{cases}\mathbb{Z}/3, & k=1,3\\ 0, & k=0,2,4.\end{cases}$$

4.5.3.1. *Class D, spinless case.* As usual in the spinless case for unorientable representations, the ansatz asks us to let $X\coloneqq(BS_4)^{3-V_\lambda}$ and consider $\mathit{MTSpin}\wedge X$.

Theorem 4.5.39. *The first few spin bordism groups of X are*

$$\begin{aligned}\widetilde{\Omega}_0^{\mathrm{Spin}}(X)&\cong\mathbb{Z}/2\\ \widetilde{\Omega}_1^{\mathrm{Spin}}(X)&\cong\mathbb{Z}/3\\ \widetilde{\Omega}_2^{\mathrm{Spin}}(X)&\cong\mathbb{Z}/2\\ \widetilde{\Omega}_3^{\mathrm{Spin}}(X)&\cong\mathbb{Z}/2\\ \widetilde{\Omega}_4^{\mathrm{Spin}}(X)&\cong\mathbb{Z}/4\oplus(\mathbb{Z}/2)^{\oplus 2},\end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}(X)$ is torsion,

Proof. For odd-primary information, see (4.5.38a). For 2-primary information, we will again use the Adams spectral sequence over $\mathcal{A}(1)$. Our first task is to write down $\widetilde{H}^*(X;\mathbb{Z}/2)$ as an $\mathcal{A}(1)$-module in low degrees, using Proposition 4.5.33 to deduce $w_1(3-V_\lambda)=a$ and $w_2(3-V_\lambda)=a^2+b$. We describe this $\mathcal{A}(1)$-module structure in low degrees in Figure 25, left.

Let Σ^4N_2 denote the submodule generated by Ub^2 and Ubc, which is a nontrivial extension of J by ΣJ.[27] Then there is an isomorphism

(4.5.40)
$$\widetilde{H}^*(X;\mathbb{Z}/2)\cong\mathcal{A}(1)\oplus\Sigma^2N_1\oplus\Sigma^4\mathcal{A}(1)\oplus\Sigma^4N_2\oplus\Sigma^6\mathcal{A}(1)\oplus P,$$

[27] We propose calling N_2 the *butterfly*; it also appears in [WWZ20, Figure 16].

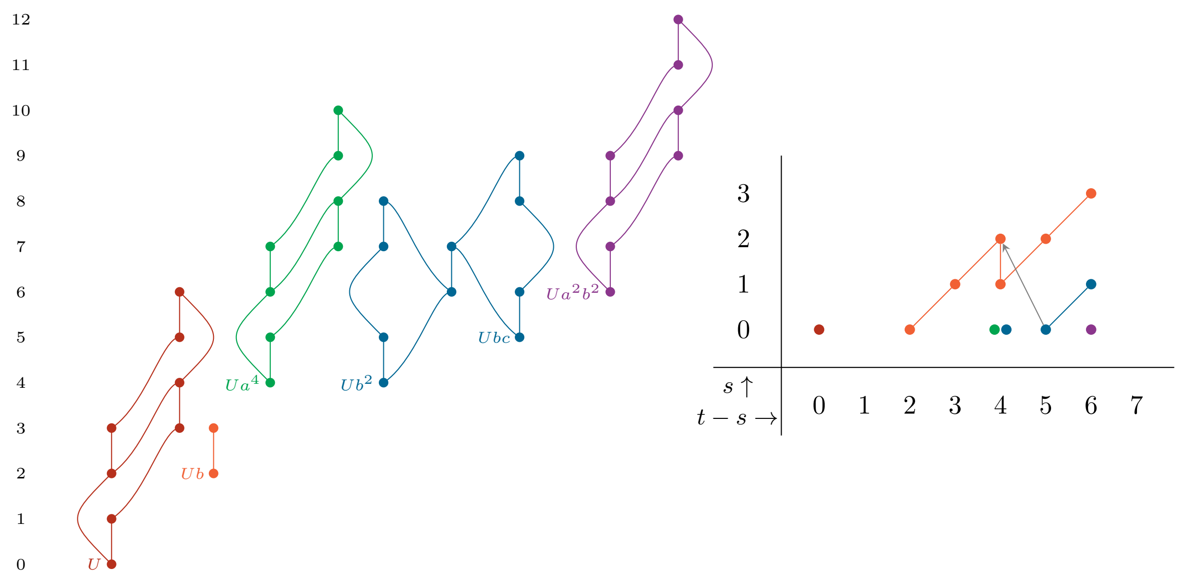


FIGURE 25. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BS_4)^{3-V_\lambda};\mathbb{Z}/2)$ in low degrees. This submodule contains all elements of degree at most 7. Right: the E_2-page of the Adams spectral sequence computing $\widetilde{ko}_*((BS_4)^{3-V_\lambda})$.

The indecomposable summand isomorphic to $\Sigma^2 N_1$ is generated by Ub, and P has no elements in degrees below 8, and therefore is irrelevant for our low-degree computations. As before, we know what a $\Sigma^k\mathcal{A}(1)$ summand contributes to the E_2-page. To compute $\mathrm{Ext}(N_1)$, we use a well-known explicit (12-shifted) 4-periodic minimal resolution[28]

$$N_1 \xleftarrow{f_0} \mathcal{A}(1) \xleftarrow{f_1} \Sigma^2\mathcal{A}(1)\oplus\Sigma^3\mathcal{A}(1) \xleftarrow{f_2} \Sigma^4\mathcal{A}(1)\oplus\Sigma^5\mathcal{A}(1) \xleftarrow{f_3} \Sigma^7\mathcal{A}(1) \xleftarrow{f_4} \Sigma^{12}\mathcal{A}(1) \xleftarrow{\Sigma^{12}f_1} \dots \tag{4.5.41}$$

The dimension of $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(N_1,\mathbb{Z}/2)$ is the number of summands of $\Sigma^t\mathcal{A}(1)$ in the s^{th} module in the extension.[29] This (shifted up by 2 for $\Sigma^2 N_1$) gives the orange summands in Figure 25, right.

For $\Sigma^4 N_2$, we use a convenient shortcut: the kernel of the map f_2 in (4.5.41) is isomorphic to $\Sigma^4 N_2$. Thus, the sequence (4.5.41) except for the first two terms forms a minimal resolution for $\Sigma^4 N_2$, so for every

[28] After some practice with $\mathcal{A}(1)$-modules, writing this minimal resolution down is straightforward, if a little tedious; we found it a helpful exercise when learning this material and the interested reader might too. Though this minimal resolution is certainly known, it is not explicitly written in many places; the resolution will not be televised.

[29] The $H^{*,*}(\mathcal{A}(1))$-action on $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(N_1,\mathbb{Z}/2)$ is a little obscure from this perspective; one can show that all h_0- and h_1-actions that could be nonzero for degree reasons are in fact nonzero, as stated in [**BB96**, §3] and [**WWZ20**, Figure 15]. One way to see this would be to use the long exact sequences in Ext associated to the two short exact sequences

$$0 \longrightarrow \Sigma\mathbb{Z}/2 \longrightarrow N_1 \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \tag{4.5.42a}$$

$$0 \longrightarrow \Sigma^2 N_1 \longrightarrow \mathcal{\mathring{O}} \longrightarrow \mathbb{Z}/2 \longrightarrow 0, \tag{4.5.42b}$$

together with the fact that the boundary maps in the long exact sequences commute with the $H^{*,*}(\mathcal{A}(1))$-action.

$s, t \geq 0$, there is an isomorphism

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(N_2, \mathbb{Z}/2) \cong \mathrm{Ext}^{s+2,t+4}_{\mathcal{A}(1)}(N_1, \mathbb{Z}/2) \tag{4.5.43}$$

equivariant for the $H^{*,*}(\mathcal{A}(1))$-actions on both sides. This gives us the blue summands in Figure 25, right. Now we can draw the E_2-page for the Adams spectral sequence for $\widetilde{\Omega}^{\mathrm{Spin}}_*(X)$, and do so in Figure 25, right.

Margolis' theorem and h_i-equivariance of differentials imply there is a single differential in this range that could be nonzero, namely the pictured $d_2\colon E_2^{0,5} \to E_2^{2,6}$.

Proposition 4.5.44. *$d_2\colon E_2^{0,5} \to E_2^{2,6}$ vanishes; equivalently, $\widetilde{ko}_4(X)$ has more than eight elements.*

We will prove this using the Atiyah-Hirzebruch spectral sequence in Theorem 4.5.53.

Assuming Proposition 4.5.44 for now, there are no further differentials in the range we care about, but we must address four extension questions in degrees 4, 5, and 6:

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow A \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow 0 \tag{4.5.45a}$$

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \widetilde{ko}_4(X) \longrightarrow A \longrightarrow 0 \tag{4.5.45b}$$

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \widetilde{ko}_5(X) \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \tag{4.5.45c}$$

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \widetilde{ko}_6(X) \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow 0. \tag{4.5.45d}$$

(In fact, a priori, there are five extension problems, but Margolis' theorem splits $E_\infty^{0,6} \cong \mathbb{Z}/2$ off from the rest of the $t - s = 6$ line.)

Both (4.5.45a) and (4.5.45c) split for the same reason. For $k = 4, 5$, assume the sequence does not split; then, $\widetilde{ko}_k(X)$ has an element x such that $2x \neq 0$ and if y is the image of $2x$ in the E_∞-page, then $h_1 y \neq 0$. This fact lifts to a nonzero action by $\eta \in ko_1$ carrying $2x$ to some element $z \in \widetilde{ko}_{k+1}(X)$ such that $z = 2\eta x$ and $z \neq 0$, but $2\eta = 0$, causing a contradiction.

Because (4.5.45a) splits and $(h_0\cdot)\colon E_\infty^{1,5} \to E_\infty^{2,6}$ is an isomorphism, all possible extensions in (4.5.45b) give $\widetilde{ko}_4(X) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 2}$.

Lastly, (4.5.45d). Action by h_1 defines isomorphisms $E_\infty^{0,5} \to E_\infty^{1,7}$ and $E_\infty^{2,7} \to E_\infty^{3,9}$, and this lifts to imply $(\eta\cdot)\colon \widetilde{ko}_6(X) \to \widetilde{ko}_7(X)$ is injective, splitting (4.5.45d). □

We return to Proposition 4.5.44. Our proof strategy is to compute $\widetilde{ko}_4(X)$ a different way. First, we pass to $\tau_{0:4}ko$-cohomology, following a strategy of Campbell [**Cam17**, §7.4] and Freed-Hopkins [**FH19a**, §5.1], by way of Lemma 4.5.46. We then run the Atiyah-Hirzebruch spectral sequence computing the $\tau_{0:4}ko$-cohomology of X. As input, we need $\widetilde{H}^*(X;\mathbb{Z})$, which we compute in Theorem 4.5.47 using a Lyndon-Hochschild-Serre spectral sequence [**Lyn48, Ser50, HS53**].

Lemma 4.5.46 (Campbell [**Cam17**, (7.35), (7.36)]). *There is a noncanonical equivalence $I_{\mathbb{Z}}(\tau_{0:4}ko) \simeq \Sigma^{-4}\tau_{0:4}ko$. Thus, if $\tau_{0:4}\widetilde{ko}_k(Y)$ is torsion, $\tau_{0:4}\widetilde{ko}_k(Y) \cong \tau_{0:4}\widetilde{ko}^{k-3}(Y)$.*

This is a corollary of the shifted self-equivalence $I_{\mathbb{Z}}KO \simeq \Sigma^4 KO$ [**And69**, Theorem 4.16].[30]

By Lemma 4.3.16, $\widetilde{ko}_4(X) \cong \tau_{0:4}\widetilde{ko}_4(X)$[31] is torsion, so is isomorphic to $\tau_{0:4}\widetilde{ko}^1(X)$. We study this group with the Atiyah-Hirzebruch spectral sequence. As input, we compute $\widetilde{H}^*(X;\mathbb{Z}_{(2)})$, which the Thom isomorphism equates with $H^*(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)})$.

Theorem 4.5.47.

$$H^0(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)}) \cong 0$$
$$H^1(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)}) \cong \mathbb{Z}/2$$
$$H^2(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)}) \cong 0$$
$$H^3(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$
$$H^4(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)}) \cong \mathbb{Z}/2$$
$$H^5(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Proof. Let $R \coloneqq \mathbb{Z}_{(2)}[x]/(x^2-1)$, which is a $\mathbb{Z}[C_2]$-module in which the nontrivial element of C_2 sends $1 \mapsto 1$ and $x \mapsto -x$. As $\mathbb{Z}[C_2]$-modules, $R \cong \mathbb{Z}_{(2)} \oplus (\mathbb{Z}_{(2)})_\sigma$, so we will recover $H^*(BS_4;(\mathbb{Z}_{(2)})_{w_1(V_\lambda)})$ from $H^*(BS_4;R)$. The Lyndon-Hochschild-Serre spectral sequence

$$E_2^{*,*} = H^*(BC_2;H^*(BA_4;R)) \Longrightarrow H^*(BS_4;R) \tag{4.5.48}$$

is multiplicative; here S_4 acts on R through $sign\colon S_4 \to C_2$ and A_4 acts trivially. R is a $\mathbb{Z}/2$-graded ring, where x is in odd degree, and hence R-valued cohomology is $\mathbb{Z}\times\mathbb{Z}/2$-graded. We use $\{+,-\}$ to denote the $\mathbb{Z}/2$-grading.

Proposition 4.5.49 (Čadek [**Čad99**, Lemma 3.1]). *There is an isomorphism of $\mathbb{Z}\times\mathbb{Z}/2$-graded rings $H^*(BC_2;R) \cong \mathbb{Z}_{(2)}[y]/(2y)$ with $|y| = (1,-)$.*

[30]Anderson gives this proof in unpublished lecture notes; see Yosimura [**Yos75**, Theorem 4] for Anderson's proof. There are at least four additional proofs that $I_{\mathbb{Z}}KO \simeq \Sigma^4 KO$, due to Freed-Moore-Segal [**FMS07**, Proposition B.11], Heard-Stojanoska [**HS14**, Theorem 8.1], Ricka [**Ric16**, Corollary 5.8], and Hebestreit-Land-Nikolaus [**HLN20**, Example 2.8], all by different methods.
[31]We abuse notation slightly to let $\tau_{0:4}\widetilde{ko}$ denote reduced $\tau_{0:4}ko$-cohomology, rather than $\widetilde{\tau_{0:4}ko}$.

Proposition 4.5.50 (Bruner-Greenlees [**BG03**, §2.6])**.** *There is a presentation of $H^*(BA_4;\mathbb{Z}_{(2)})$ whose only generators and relations below degree 6 are generators α and β in degrees 3 and 4, respectively, and relations $2\alpha = 2\beta = 0$.*

Corollary 4.5.51. *As $\mathbb{Z}\times\mathbb{Z}/2$-graded rings,*

$$H^*(BA_4;R) \cong \mathbb{Z}_{(2)}[\alpha_+,\alpha_-,\beta_+,\beta_-,\dots]/(2\alpha_\pm, 2\beta_\pm,\dots) \tag{4.5.52}$$

where the generators and relations not displayed are in $\mathbb{Z}$-degrees ≥ 6, $|\alpha_\pm| = (2,\pm)$, and $|\beta_\pm| = (3,\pm)$.

We now display the E_2-page in Figure 26. Elements with $+$ grading are colored red, and elements with $-$ grading are colored blue; differentials are even in this $\mathbb{Z}/2$-grading. The map $S_4\to C_2$ admits a section given

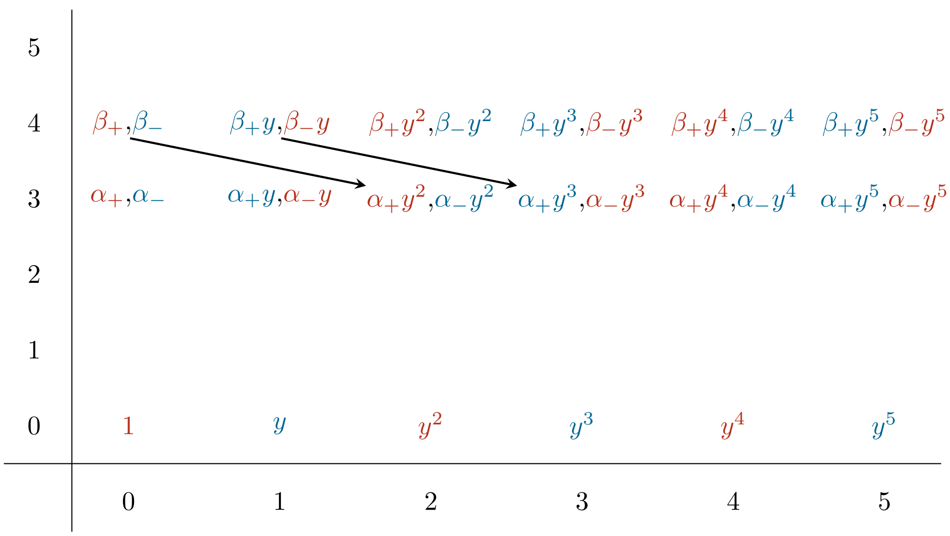


FIGURE 26. The Lyndon-Hochschild-Serre spectral sequence computing $H^*(BS_4;R)$. Colors represent a $\mathbb{Z}/2$-grading induced from a $\mathbb{Z}/2$-grading on R.

by $\{1,(1\ 2)\}\subset S_4$, so the $q=0$ line supports no nonzero differentials and does not participate in nontrivial extension problems. Looking just at elements graded $-$, we are done if we can show that $d_2(\beta_-) = \alpha_-y^2$ and $d_2(\beta_+y) = 0$. Fortunately, Thomas [**Tho74**] has computed $H^*(BS_4;\mathbb{Z}_{(2)})$: since $H^4(BS_4;\mathbb{Z}_{(2)})\cong\mathbb{Z}/4\oplus\mathbb{Z}/2$, $d_2(\beta_+) = 0$, so the Leibniz rule implies $d_2(\beta_+y) = 0$ too. And since $H^5(BS_4;\mathbb{Z}_{(2)})\cong\mathbb{Z}/2$, $d_2(\beta_-y)\neq 0$, so $d_2(\beta_-)\neq 0$, hence must be α_-y^2. ☐

Thus equipped, we tackle the Atiyah-Hirzebruch spectral sequence.

Theorem 4.5.53. $|\widetilde{ko}_4(X)|\geq 16$ *(thus implying Proposition 4.5.44).*

Proof. After using Lemma 4.5.46, we want to compute $\tau_{0:4}\widetilde{ko}^1(X)$, which we attack with the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = \widetilde{H}^p(X; (\tau_{0:4}ko)^q) \Longrightarrow \tau_{0:4}\widetilde{ko}^{p+q}(X). \tag{4.5.54}$$

Using Proposition 4.5.32 and Theorem 4.5.47 as input, the E_2-page is Maunder [**Mau63**, Theorem 3.4]

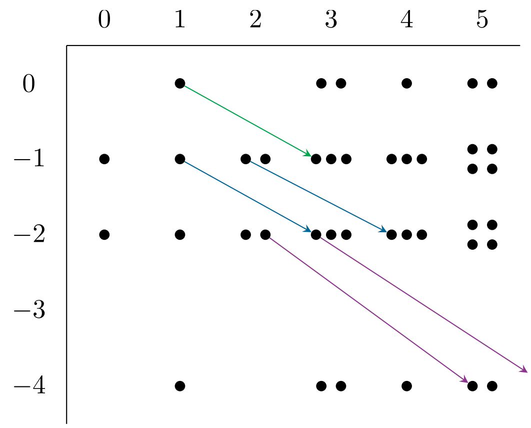


Figure 27. The Atiyah-Hirzebruch spectral sequence computing $\tau_{0:4}\widetilde{ko}^*(X)$.

identifies the first nonzero differential in the cohomological Atiyah-Hirzebruch spectral sequence with a k-invariant; this includes all differentials shown in Figure 27. Let $r\colon H^*(-;\mathbb{Z}) \to H^*(-;\mathbb{Z}/2)$ denote reduction mod 2 and $\beta\colon H^*(-;\mathbb{Z}/2) \to H^{*+1}(-;\mathbb{Z})$ be the Bockstein. Then, Bruner-Greenlees [**BG10**, Corollary A.5.2] determine the k-invariants we need for ko-cohomology:

(1) The green $d_2\colon E_2^{p,0} \to E_2^{p+2,-1}$ is $\mathrm{Sq}^2 \circ r\colon \widetilde{H}^p(X;\mathbb{Z}) \to \widetilde{H}^{p+2}(X;\mathbb{Z}/2)$.
(2) Each blue $d_2\colon E_2^{p,-1} \to E_2^{p+2,-1}$ is $\mathrm{Sq}^2\colon \widetilde{H}^p(X;\mathbb{Z}/2) \to \widetilde{H}^{p+2}(X;\mathbb{Z}/2)$.
(3) Each purple $d_3\colon E_3^{p,-2} \to E_3^{p+3,-4}$ is $\beta \circ \mathrm{Sq}^2\colon \widetilde{H}^p(X;\mathbb{Z}/2) \to \widetilde{H}^{p+3}(X;\mathbb{Z})$.

We computed the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X;\mathbb{Z}/2)$ in (4.5.40) (and drew it in Figure 25, left), and r and β follow from this and a few facts we just calculated for $\widetilde{H}^*(X;\mathbb{Z})$. For $k \le 5$, we proved $2\widetilde{H}^k(X;\mathbb{Z}) = 0$, so r is injective in these degrees. Moreover, combining this with Lemma 4.3.20, that $r \circ \beta = \mathrm{Sq}^1$, we conclude for $k \le 2$ and $x \in \widetilde{H}^k(X;\mathbb{Z}/2)$, $\beta\mathrm{Sq}^2(x) = 0$ iff $\mathrm{Sq}^1\mathrm{Sq}^2(x) = 0$.

All together, these allow us to resolve almost all of the indicated differentials — a priori, we do not know $\beta\mathrm{Sq}^2(x)$ when $x \in E_2^{3,-2} \cong \widetilde{H}^3(X;\mathbb{Z}/2)$, but for all x not in the image of $d_2\colon E_2^{1,-1} \to E_2^{3,-2}$, $\mathrm{Sq}^2(x) = 0$, so this is fine. We find the 1-line of the E_4-page has five $\mathbb{Z}/2$ summands, one in $E_4^{2,-1}$, two in $E_4^{3,-2}$, and two in $E_4^{5,-4}$. There could be a nonzero $d_4\colon E_4^{2,-1} \to E_4^{6,-5}$, but the remaining four summands are generated by permanent cycles. $\square$

4.5.3.2. *Class D, spin-1/2 case.* As V_λ is not pin$^-$, Theorem 4.2.11 tells us to compute the spin bordism of $X := (BS_4)^{\mathrm{Det}(V_\lambda)-1}$.

Theorem 4.5.55. *The first few spin bordism groups of X are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_1^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/6\\
\widetilde{\Omega}_2^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/8\oplus\mathbb{Z}/2\\
\widetilde{\Omega}_3^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_4^{\mathrm{Spin}}(X) &\cong 0,
\end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}(X)$ is torsion.

Proof. Odd-primary information is computed in the range we need by (4.5.38a). For 2-primary information, we use the Adams spectral sequence as usual. Recall the $\mathcal{A}(1)$-module structure on $H^*(BS_4;\mathbb{Z}/2)\cong \mathbb{Z}/2[a,b,c]/(ac)$ from Propositions 4.5.32 and 4.5.33. Lemma 4.3.22 shows that inclusion of a transposition extends to a splitting

$$X \xrightarrow{\simeq} (B\mathbb{Z}/2)^{\sigma-1}\vee M, \tag{4.5.56}$$

and the map $\widetilde{H}^*(M;\mathbb{Z}/2)\to\widetilde{H}^*(X;\mathbb{Z}/2)$ is injective, with image a complementary subspace to the span of $\{Ua^n\mid n\ge 0\}$. As usual, we write down $\widetilde{H}^*(M;\mathbb{Z}/2)$ as an $\mathcal{A}(1)$-module in low degrees, using $w_1(\mathrm{Det}(V_\lambda)-1)=a$ and $w_2(\mathrm{Det}(V_\lambda)-1)=0$, and give the answer in Figure 28, left.

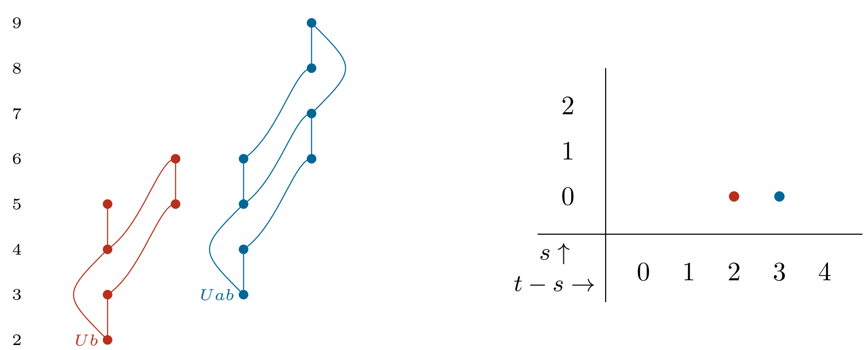


Figure 28. Left: The $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ in low degrees. This submodule contains all elements of degree at most 4. Right: the Ext of this module, which is the beginning of the Adams spectral sequence computing $\widetilde{ko}_*(M)$. More information in the proof of Theorem 4.5.55.

Let $\Sigma^2 N_3$ denote the $\mathcal{A}(1)$-submodule generated by Ub; this module is studied by Baker [**Bak18**, §5], who calls it the "whiskered Joker." There is an isomorphism of $\mathcal{A}(1)$-modules

$$\widetilde{H}^*(M;\mathbb{Z}/2) \cong \Sigma^2 N_3 \oplus \Sigma^3\mathcal{A}(1) \oplus P, \tag{4.5.57}$$

where P contains no elements of degree less than 4. Therefore if the 4-line of the E_2-page is empty, P does not enter into our calculations — and we will see momentarily that the 4-line is in fact empty. We know what $\Sigma^3\mathcal{A}(1)$ summand contributes to the E_2-page of the Adams spectral sequence. For N_3, we leverage what we learned from N_1 in §4.5.3.1. Specifically, the unique nonzero $\mathcal{A}(1)$-module map $\mathcal{A}(1) \to N_3$ has kernel isomorphic to $\Sigma^5 N_1$, so a minimal resolution for $\Sigma^5 N_1$ induces a minimal resolution for N_3 which has an additional copy of $\mathcal{A}(1)$ in topological degree 0 and filtration 0, and in which everything else is shifted up one in filtration, giving the red summands in Figure 28, right.

Thus the E_2-page for this Adams spectral sequence is as in Figure 28, right. In this range, the spectral sequence collapses. Combine this with the pin$^-$ bordism summands from [**ABP69, KT90b**] as usual to obtain the groups in the theorem statement, and Lemma 4.3.16 finishes us off by telling us $\widetilde{\Omega}_5^{\mathrm{Spin}}(X)$ is torsion. $\square$

4.5.3.3. *Class A, spinless case.* Let f_0^A denote the equivariant local system of symmetry types for class A with spinless fermions. In this case, the ansatz asks us to consider the spin^c bordism of $X \coloneqq (BS_4)^{3-V_\lambda}$.

Theorem 4.5.58. *The first few spin^c bordism groups of X are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X) &= \mathbb{Z}/2\\
\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X) &= \mathbb{Z}/3\\
\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X) &= (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X) &= \mathbb{Z}/3\\
\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X) &= (\mathbb{Z}/2)^{\oplus 4},
\end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X)$ is torsion. Therefore $Ph_0^{S_4}(\mathbb{R}^3, f_0^A) \cong (\mathbb{Z}/2)^{\oplus 4}$.

Proof. We will use the Adams spectral sequence over $\mathcal{E}(1)$ as usual to capture the 2-primary information; for odd-primary information, see (4.5.38b).

We use the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(X;\mathbb{Z}/2)$ that we determined in (4.5.40) and drew in Figure 25 to determine the $\mathcal{E}(1)$-module structure: as $\mathcal{E}(1)$-modules, $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$, and $N_2 \cong \mathcal{E}(1) \oplus \Sigma\mathcal{E}(1) \oplus \Sigma^2 N_1$, so as $\mathcal{E}(1)$-modules,

$$\widetilde{H}^*(X;\mathbb{Z}/2) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1) \oplus \Sigma^2 N_1 \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^5\mathcal{E}(1) \oplus P, \tag{4.5.59}$$

where P is 5-connected. We draw this in Figure 29, left. Recalling $\mathrm{Ext}_{\mathcal{E}(1)}(N_1)$ from (4.4.50), the E_2-page

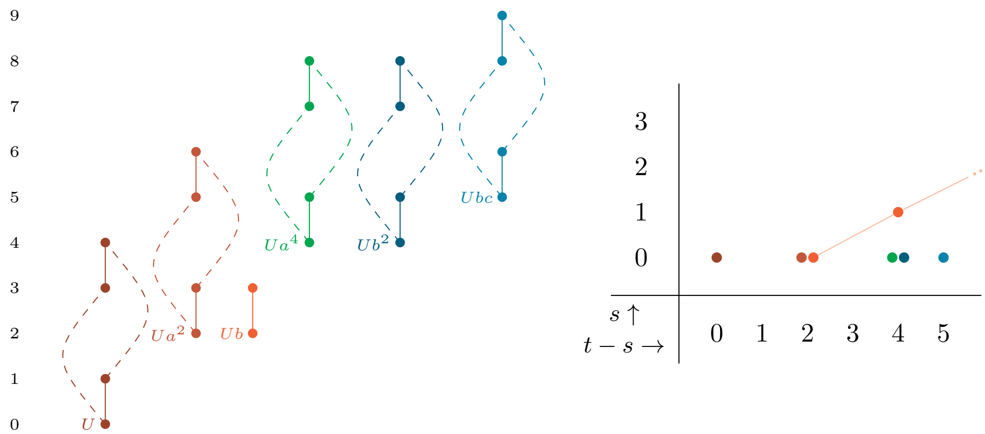


FIGURE 29. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*((BS_4)^{3-V_\lambda};\mathbb{Z}/2)$ in low degrees. The pictured submodule contains all elements of degree at most 5. Right: the Ext of this module, which is the beginning of the E_2-page of the Adams spectral sequence computing $\widetilde{ku}_*((BS_4)^{3-V_\lambda})$.

of the Adams spectral sequence is in Figure 29, right. There can be no differentials in the range drawn for degree reasons, and Margolis' theorem (Theorem 4.3.14) implies there are no nontrivial extensions, either, so we are done. $\square$

4.5.3.4. *Class A, spin-1/2 case.* Theorem 4.2.24 says that to compute the S_4-equivariant phase homology groups in class A with spin-1/2 fermions, given by the equivariant local system of symmetry types $f^A_{1/2}$, we should investigate the spin^c bordism of $X := (BS_4)^{\mathrm{Det}\,V_\lambda - 1}$: we know V_λ is not pin^c because its pullback along $BA_4 \to BS_4$ is not pin^c, as we established in Lemma 4.5.7.

Theorem 4.5.60. *The first few spin^c bordism groups of X are*

$$\widetilde{\Omega}_0^{\mathrm{Spin}^c}(X) \cong \mathbb{Z}/2$$
$$\widetilde{\Omega}_1^{\mathrm{Spin}^c}(X) \cong \mathbb{Z}/3$$
$$\widetilde{\Omega}_2^{\mathrm{Spin}^c}(X) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$$
$$\widetilde{\Omega}_3^{\mathrm{Spin}^c}(X) \cong \mathbb{Z}/6$$
$$\widetilde{\Omega}_4^{\mathrm{Spin}^c}(X) \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 2}.$$

By Lemma 4.3.16, $\widetilde{\Omega}_5^{\mathrm{Spin}^c}(X)$ is torsion, so $Ph_0^{S_4}(\mathbb{R}^3; f_{1/2}^A) \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 2}$.

Proof. See (4.5.38b) for the odd-primary torsion in $\widetilde{\Omega}_*^{\mathrm{Spin}^c}(X)$. For 2-torsion, we reuse our work from §4.5.3.2. First, $X \simeq (B\mathbb{Z}/2)^{\sigma-1} \vee M$, and we gave the low-degree cohomology of M as an $\mathcal{A}(1)$-module in (4.5.57), and drew it in Figure 28, left. This determines the $\mathcal{E}(1)$-module structure on it, so we can calculate spin^c bordism of M using the Adams spectral sequence. For the other summand, we have $\mathit{MTSpin}^c \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq \mathit{MTPin}^c$, so we direct-sum in the pin^c bordism groups computed by Bahri-Gilkey [**BG87a, BG87b**].

There are isomorphisms of $\mathcal{E}(1)$-modules $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$ and $N_3 \cong \mathcal{E}(1) \oplus \Sigma^2 N_1$. Therefore as an $\mathcal{E}(1)$-module,

$$\widetilde{H}^*(M;\mathbb{Z}/2) \cong \Sigma^2\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^4 N_1 \oplus P, \tag{4.5.61}$$

where P is 4-connected. As usual for these cases, we will see that $\mathrm{Ext}(\widetilde{H}^*(M;\mathbb{Z}/2),\mathbb{Z}/2)$ has no nonzero elements with $t-s=4$ and $s>1$, so P does not affect our calculations. See Figure 30, left, for a picture of the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$. We calculated $\mathrm{Ext}(\Sigma^4 N_1)$ in (4.4.50), so can draw the E_2-page of

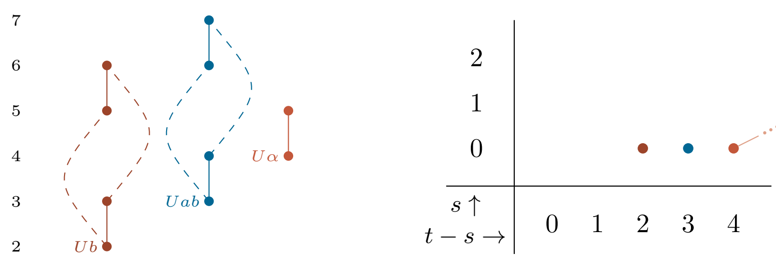


Figure 30. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ in low degrees; the pictured summands include all elements in degrees 4 and below. Here $\alpha := a^2b + b^2$. Right: the Ext of this module, which is the beginning of the E_2-page of the Adams spectral sequence computing $\widetilde{ku}_*(M)$.

the Adams spectral sequence in Figure 30, right. This collapses, so we add in the pin^c bordism summands and conclude. $\square$

4.5.4. Chiral octahedral symmetry. Let $\lambda\colon S_4 \to \mathrm{O}_3$ denote the representation as symmetries of an octahedron and $V_\lambda \to BS_4$ denote the associated vector bundle. Recall from Proposition 4.5.32 the mod 2 cohomology of BS^4.

Lemma 4.5.62. *$w_1(V_\lambda) = 0$ and $w_2(V_\lambda) = b$.*

Proof. Since $\mathrm{Im}(\lambda) \subset \mathrm{SO}_3$, $w_1(V_\lambda) = 0$. We know $w_2(V_\lambda)$ restricts to $u \in H^2(BA_4;\mathbb{Z}/2)$ by considering tetrahedral symmetry inside octahedral symmetry and using Lemma 4.5.3, so $w_2(V_\lambda)$ could be $a^2 + b$ or b.

The fact that λ splits as $\sigma \oplus \mathbb{R}^2$ when restricted to a $\mathbb{Z}/2$ subgroup given by a transposition tells us $w_2(V_\lambda)$ is b, not $a^2 + b$. ☐

By Lemma 4.5.7, the pullback of V_λ to BA_4 is not pin^c, so V_λ is not pin^c, and hence V_λ is also not pin$^-$.

Lemma 4.5.63.

$$\widetilde{\Omega}_k^{\mathrm{SO}}(BS_4) \otimes \mathbb{Z}[1/2] \cong \begin{cases} \mathbb{Z}/3, & k = 3 \\ 0, & k = 0, 1, 2, 4, 5, 6. \end{cases} \tag{4.5.64}$$

Proof. Let ℓ be an odd prime and consider the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(BS_4; (MTSO^\wedge_\ell)_q) \Longrightarrow (MTSO^\wedge_\ell)_{p+q}(BS_4) = \Omega^{\mathrm{SO}}_{p+q}(BS_4)^\wedge_\ell. \tag{4.5.65}$$

If $\ell \neq 3$, then $\ell \nmid |S_4|$, so the $\mathbb{Z}_\ell$-cohomology of BS_4 vanishes in positive degrees and (4.5.65) is trivial, contributing no ℓ-torsion to $\widetilde{\Omega}_*^{\mathrm{SO}}(BS_4) \otimes \mathbb{Z}[1/2]$. For $\ell = 3$, use Thomas' calculation of $H^*(BS_4; \mathbb{Z})$ [Tho74] and the universal coefficient theorem to show that $H_*(BS_4; \mathbb{Z}_3)$ consists of $\mathbb{Z}_3$ in degree 0, $\mathbb{Z}/3$ in degree 2, and nothing else nonzero in degrees 5 and below. Therefore (4.5.65) collapses, giving us the desired result. ☐

4.5.4.1. *Class D, spinless case.* Let f_0^D denote the equivariant local system of symmetry types for the spinless class D case. Theorem 4.2.11 identifies

$$Ph_k^{S_4}(\mathbb{R}^3; f_0^D) \cong [MTSpin \wedge (BS_4)^{3 - V_\lambda}, \Sigma^{k+4} I_{\mathbb{Z}}], \tag{4.5.66}$$

so we study the spin bordism of $X \coloneqq (BS_4)^{3 - V_\lambda}$.

Theorem 4.5.67. *There is an $r \geq 2$ such that the first few spin bordism groups of X are*

$$\begin{aligned} \widetilde{\Omega}_0^{\mathrm{Spin}}(X) &\cong \mathbb{Z} \\ \widetilde{\Omega}_1^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/2 \\ \widetilde{\Omega}_2^{\mathrm{Spin}}(X) &\cong 0 \\ \widetilde{\Omega}_3^{\mathrm{Spin}}(X) &\cong \mathbb{Z}/6 \oplus \mathbb{Z}/2^k \\ \widetilde{\Omega}_4^{\mathrm{Spin}}(X) &\cong \mathbb{Z}, \end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}(X)$ is torsion. Hence $Ph_0^{S_4}(\mathbb{R}^3; f_0^D) = 0$.

The Atiyah-Hirzebruch spectral sequence allows one to show $k = 1$, so $\widetilde{\Omega}_3^{\mathrm{Spin}}(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. As usual, we will not need this, so do not prove it.

Proof. For odd-primary torsion, use the fact that $MTSpin \to MTSO$ is an isomorphism, so it suffices to understand $\widetilde{\Omega}_*^{\mathrm{SO}}(X)$, and that $V_\lambda \to BS_4$ is orientable, so there is a Thom isomorphism $\Omega_k^{\mathrm{SO}}(BS_4) \to \widetilde{\Omega}_k^{\mathrm{SO}}(X)$, and we can read off the odd-primary torsion from Lemma 4.5.63.

On to the prime 2. From Propositions 4.5.32 and 4.5.33 we know the mod 2 cohomology of BS_4 and the action of the Steenrod algebra, and using Lemma 4.5.62 we can draw $\widetilde{H}^*(X;\mathbb{Z}/2)$ as an $\mathcal{A}(1)$-module in low degrees, which we do in Figure 31, left.

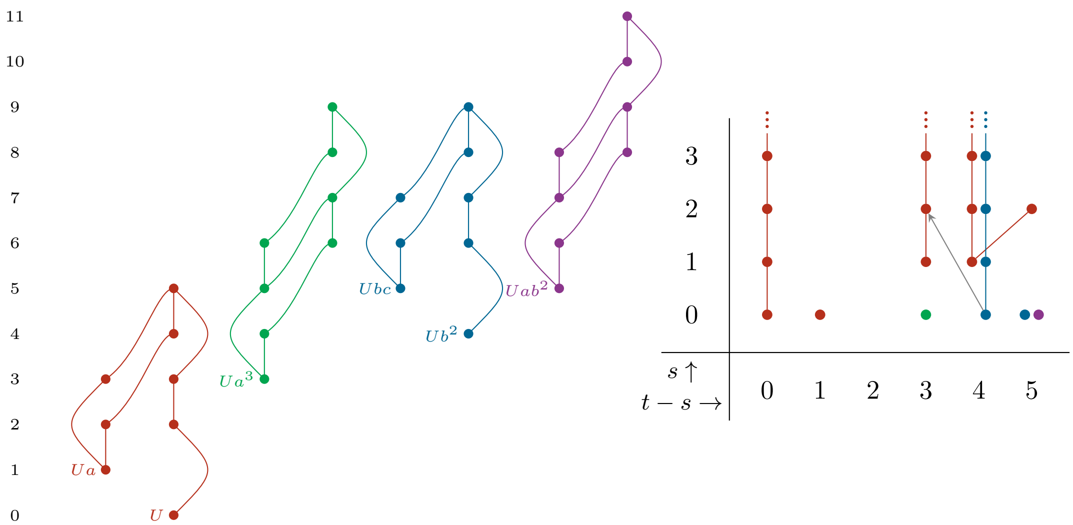


Figure 31. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((BS_4)^{3-V_\lambda};\mathbb{Z}/2)$ in low degrees. The pictured submodule contains all elements of degrees 6 and below. Right: the E_2-page of the corresponding Adams spectral sequence computing $\widetilde{ko}_*((BS_4)^{3-V_\lambda})_2^\wedge$. We will see there is a differential from the 4-line to the 3-line; it is in fact the d_2 depicted, though we do not prove that.

Let N_4 denote the $\mathcal{A}(1)$-submodule of $\widetilde{H}^*(X;\mathbb{Z}/2)$ generated by U and Ua. Then,

$$\widetilde{H}^*(X;\mathbb{Z}/2) \cong N_4 \oplus \Sigma^3\mathcal{A}(1) \oplus \Sigma^4 N_4 \oplus \Sigma^5\mathcal{A}(1) \oplus P, \tag{4.5.68}$$

where P is 6-connected. We have not seen N_4 before, and need to calculate its Ext. Fortunately, there is a short exact sequence of $\mathcal{A}(1)$-modules

$$0 \longrightarrow \Sigma J \longrightarrow N_4 \longrightarrow \widetilde{\mathcal{O}} \longrightarrow 0, \tag{4.5.69}$$

which induces a long exact sequence in Ext. In Figure 32, we display a picture both of this extension and of the Adams chart for computing the boundary map in the long exact sequence.

We draw the E_2-page in Figure 31, right. Because differentials must be h_0-equivariant, they all vanish in the range pictured except possibly for those from the 4-line to the 3-line, one of which is indicated in

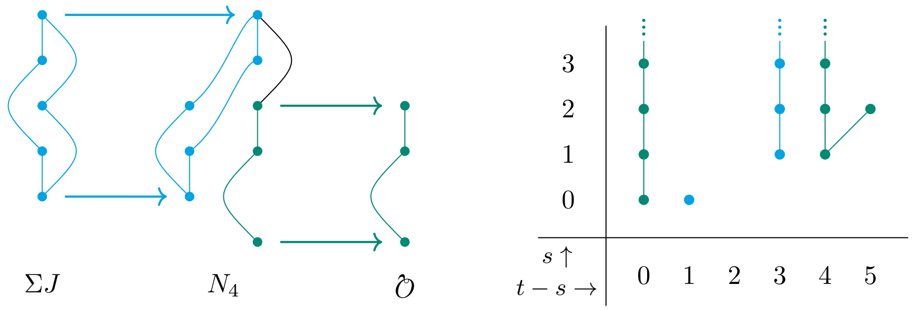


FIGURE 32. Left: the extension (4.5.69) of $\mathcal{A}(1)$-modules. Right: the long exact sequence in Ext induced from that extension.

the chart. By Lemma 4.3.16, $\widetilde{ko}_4(X)\otimes\mathbb{Q}\cong\widetilde{ko}_0(X)\otimes\mathbb{Q}$, and from Figure 31, right, the latter group is isomorphic to $\mathbb{Q}$. Thus $\widetilde{\Omega}_4^{\mathrm{Spin}}(X)$ has exactly one free summand, so one of the two towers in the 4-line lives to the E_∞-page, and the other admits a nonzero d_r differential to the tower in degree 3. Thus, on the 3-line of the E_{r+1}-page, there is a single green $\mathbb{Z}/2$ summand in degree $s=0$, together with a red tower with finitely many $\mathbb{Z}/2$ summands, giving $\mathbb{Z}/2\oplus\mathbb{Z}/2^k$ in degree 3 as promised.[32] $\square$

4.5.4.2. *Class D, spin-1/2 case.* Let $f_{1/2}^D$ be the S_4-equivariant local system of symmetry types for the case of spin-1/2 fermions in class D. Theorem 4.2.11 computes the equivariant phase homology of this local system in terms of $\Omega_*^{\mathrm{Spin}}(BS_4)$.

Theorem 4.5.70. *There is an $r\geq 2$ such that the first several spin bordism groups of BS_4 are*

$$\Omega_0^{\mathrm{Spin}}(BS_4)\cong\mathbb{Z}$$
$$\Omega_1^{\mathrm{Spin}}(BS_4)\cong(\mathbb{Z}/2)^{\oplus 2}$$
$$\Omega_2^{\mathrm{Spin}}(BS_4)\cong(\mathbb{Z}/2)^{\oplus 3}$$
$$\Omega_3^{\mathrm{Spin}}(BS_4)\cong\mathbb{Z}/24\oplus\mathbb{Z}/2^{r+1},$$
$$\Omega_4^{\mathrm{Spin}}(BS_4)\cong\mathbb{Z}\oplus\mathbb{Z}/2$$
$$\Omega_5^{\mathrm{Spin}}(BS_4)\cong 0$$
$$\Omega_6^{\mathrm{Spin}}(BS_4)\cong\mathbb{Z}/2.$$

Therefore $Ph_0^{S_4}(\mathbb{R}^3;f_{1/2}^D)\cong\mathbb{Z}/2$.

[32] We have not determined which elements of the 4-line the differential is nonzero on. One way to determine this is to use that the generator of $H^{3,7}(\mathcal{A}(1))\cong\mathbb{Z}/2$ carries the summands in the 0-line onto a subset of the red tower in the 4-line. Differentials are equivariant for this action, and differentials emerging from the 0-line vanish, so all differentials must vanish on the red tower too.

One can use the Atiyah-Hirzebruch spectral sequence to show $r = 2$ in Theorem 4.5.70; we do not need this so do not present the proof.

PROOF. First, we use the Adams spectral sequence to determine the free and 2-primary parts. Since $ko_*(BS_4)$ splits as $ko_*(\mathrm{pt}) \oplus \widetilde{ko}_*(BS_4)$, we focus on $\widetilde{ko}_*(BS_4)$ and add the Bott-song summands in at the end. There is a section s of the parity map $S_4 \to \mathbb{Z}/2$, which stably splits BS_4. That is, there is a spectrum M, a map $t\colon M \to \Sigma^\infty BS_4$, and a weak equivalence

$$(s,t)\colon \Sigma^\infty B\mathbb{Z}/2 \vee M \xrightarrow{\simeq} \Sigma^\infty BS_4. \tag{4.5.71}$$

This also splits the $\mathcal{A}$-module structure of $\widetilde{H}^*(BS_4;\mathbb{Z}/2)$ as

$$\widetilde{H}^*(M;\mathbb{Z}/2) \oplus \widetilde{H}^*(B\mathbb{Z}/2;\mathbb{Z}/2), \tag{4.5.72}$$

where $\widetilde{H}^*(B\mathbb{Z}/2;\mathbb{Z}/2)$ is embedded via the parity map. Therefore $\widetilde{H}^*(M;\mathbb{Z}/2)$ is isomorphic to a complimentary subspace of $\mathbb{Z}/2 \cdot \{a^k \mid k \geq 0\} \subset \widetilde{H}^*(BS_4;\mathbb{Z}/2)$. As this isomorphism is realized by a map of spectra, it is an isomorphism of $\mathcal{A}$-modules, hence $\mathcal{A}(1)$-modules. We will run the Adams spectral sequence for $\widetilde{ko}_*(M)$, and add the $\widetilde{ko}_*(B\mathbb{Z}/2)$ summands in at the end.

The mod 2 cohomology of BS_4 is given in Proposition 4.5.32, and the action of the Steenrod squares in Proposition 4.5.33. We can therefore draw $\widetilde{H}^*(M;\mathbb{Z}/2)$ as an $\mathcal{A}(1)$-module in low degrees, which we do in Figure 33, left. We have

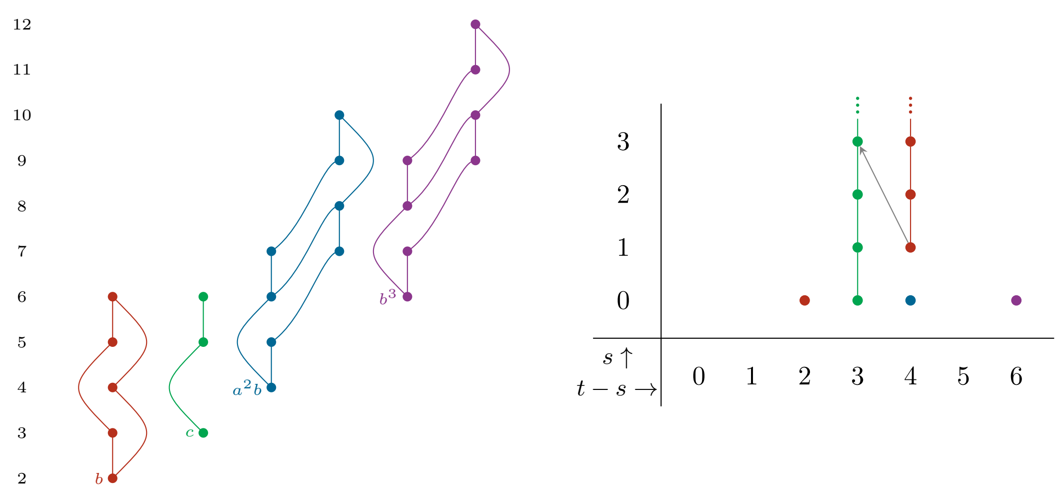


FIGURE 33. Left: the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ in low degrees. The submodule pictured here contains all elements of degree at most 6. Right: the corresponding Ext, which is the E_2-page for the Adams spectral sequence converging to the 2-primary part of $\widetilde{ko}_*(M)$.

$$(4.5.73) \qquad \widetilde{H}^*(M;\mathbb{Z}/2) \cong \Sigma^2 J \oplus \Sigma^3 \mathcal{O} \oplus \Sigma^4 \mathcal{A}(1) \oplus \Sigma^6 \mathcal{A}(1) \oplus P,$$

where P is 6-connected. Names of $\mathcal{A}(1)$-modules are as in previous sections; for all these modules except for P, we have already seen $\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(-,\mathbb{Z}/2)$, and P is irrelevant for degree reasons. We display the E_2-page in Figure 33, right.

In the range pictured, h_0-equivariance of differentials implies the only possible nontrivial differentials are from the infinite tower in degree 4 to the infinite tower in degree 3; a d_2 is pictured as an example. In fact, those towers must support a nonzero d_r for some r; by h_0-equivariance, d_r is either zero for every element of the tower in degree 4, or nonzero for every element. Hence, if all d_r were zero for all r, then $\widetilde{ko}_3(BS_4)$ would contain a free summand, contradicting Lemma 4.3.16. Therefore there is some $r \geq 2$ for which all d_r differentials from the tower in degree 4 to the tower in degree 3 are nontrivial (not necessarily the d_2s pictured). On the E_∞-page, the tower in degree 4 vanishes, and only $r+1$ summands of the degree-3 tower remain. Thus we have computed the 2-primary part of $ko_*(BS_4)$ in degrees 6 and lower:

- From $ko_*(\mathrm{pt})$, we have a $\mathbb{Z}$ summand in degrees 0 and 4 and a $\mathbb{Z}/2$ summand in degrees 1 and 2.
- From $\widetilde{ko}_*(B\mathbb{Z}/2)$, we have $\mathbb{Z}/2$ summands in degrees 1 and 2 and a $\mathbb{Z}/8$ summand in degree 3 [**MM76**].
- From Figure 33, right, we have $\mathbb{Z}/2$ in degree 2, $\mathbb{Z}/2^{r+1}$ in degree 3, and a $\mathbb{Z}/2$ each in degrees 4 and 6.

To determine the odd-primary torsion, use first that the forgetful map $\Omega_*^{\mathrm{Spin}}(-) \to \Omega_*^{\mathrm{SO}}(-)$ is an isomorphism on odd-primary torsion, so we just have to determine the odd-primary torsion in $\Omega_k^{\mathrm{SO}}(BS_4)$ for $k \leq 6$, which we did in Lemma 4.5.63. ☐

4.5.4.3. *Class A.* As in the case of chiral tetrahedral symmetry, V_λ does not admit a pin^c structure, since we saw in Lemma 4.5.7 that its pullback along $BA_4 \to BS_4$ also does not admit a pin^c structure. Let f_0^A, resp. $f_{1/2}^A$, denote the equivariant local systems of spectra associated to the class A spinless, resp. spin-1/2 cases. Theorem 4.2.24 expresses $Ph_0^{S_4}(\mathbb{R}^3; f_0^A)$ and $Ph_0^{S_4}(\mathbb{R}^3; f_{1/2}^A)$ in terms of the spin^c bordism of $(BS_4)^{3-V_\lambda}$ for spinless fermions and BS_4 for spin-1/2 fermions.

Theorem 4.5.74. *There are integers $r, r' \geq 2$ such that the low-degree spin^c bordism groups of $(BS_4)^{3-V_\lambda}$ and BS_4 are*

$$\begin{aligned}
&\widetilde{\Omega}_0^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) \cong \mathbb{Z} && \Omega_0^{\mathrm{Spin}^c}(BS_4) \cong \mathbb{Z}\\
&\widetilde{\Omega}_1^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) \cong \mathbb{Z}/2 && \Omega_1^{\mathrm{Spin}^c}(BS_4) \cong \mathbb{Z}/2\\
&\widetilde{\Omega}_2^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) \cong \mathbb{Z} && \Omega_2^{\mathrm{Spin}^c}(BS_4) \cong \mathbb{Z}\oplus\mathbb{Z}/2\\
&\widetilde{\Omega}_3^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) \cong \mathbb{Z}/6\oplus\mathbb{Z}/2^r && \Omega_3^{\mathrm{Spin}^c}(BS_4) \cong \mathbb{Z}/12\oplus\mathbb{Z}/2^{r'}\\
&\widetilde{\Omega}_4^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) \cong \mathbb{Z}^2 && \Omega_4^{\mathrm{Spin}^c}(BS_4) \cong \mathbb{Z}^2\oplus\mathbb{Z}/2\\
&\widetilde{\Omega}_5^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) \cong \mathbb{Z}/2^{r-1}\oplus\mathbb{Z}/6\oplus(\mathbb{Z}/2)^{\oplus 3} && \Omega_5^{\mathrm{Spin}^c}(BS_4) \cong \mathbb{Z}/2\oplus\mathbb{Z}/24\oplus\mathbb{Z}/2^{r'+1}\\
&\widetilde{\Omega}_6^{\mathrm{Spin}^c}((BS_4)^{3-V_\lambda}) \cong \mathbb{Z}^2 && \Omega_6^{\mathrm{Spin}^c}(BS_4) \cong \mathbb{Z}^2\oplus(\mathbb{Z}/2)^{\oplus 3}.
\end{aligned}$$

One can use the Atiyah-Hirzebruch spectral sequence to show $r = r' = 2$. We do not need this, so do not go into the details.

Proof. As usual, the calculation separates into odd-primary and 2-primary parts.

Lemma 4.5.75. *The only odd-primary torsion in the spin^c bordism of $(BS_4)^{3-V_\lambda}$ and BS_4 in degrees 6 and below consists of two $\mathbb{Z}/3$ summands in degrees 3 and 5.*

Proof. Since $|S^4| = 2^3\cdot 3$, we only have to check 3-torsion: if $\ell \geq 5$ is prime, the maps $BS_4 \to \mathrm{pt}$ and $(BS_4)^{3-V} \to \mathrm{pt}$ are stable ℓ-primary equivalences by the Whitehead theorem [**Ser53**, Chapitre III, Théorème 3]. The forgetful map $MTSpin^c \to MSO\wedge(B\mathrm{U}_1)_+$ is an odd-primary equivalence, and since $3 - V_\lambda$ is orientable, there is a Thom isomorphism

$$MSO\wedge(B\mathrm{U}_1)_+\wedge(BS_4)^{3-V} \xrightarrow{\simeq} MSO\wedge(B\mathrm{U}_1)_+\wedge(BS_4)_+, \tag{4.5.76}$$

so in both the spinless and spin-1/2 cases, we can glean the 3-torsion from $\Omega_*^{\mathrm{Spin}^c}(B\mathrm{U}_1\times BS_4)$. As the homology of $B\mathrm{U}_1$ is torsion-free, the Künneth map $H_*(B\mathrm{U}_1)\otimes H_*(BS_4) \to H_*(B\mathrm{U}_1\times BS_4)$ is an isomorphism of graded abelian groups. Using this together with Thomas' [**Tho74**] calculation of $H_*(BS_4)$, we conclude that the only odd-primary torsion in $H_*(B\mathrm{U}_1\times BS_4)$ in degrees below 7 is $\mathbb{Z}/3 \subset H_3(B\mathrm{U}_1\times BS_4)$ and $\mathbb{Z}/3 \subset H_5(B\mathrm{U}_1\times BS_4)$.

Now feed this to the Atiyah-Hirzebruch spectral sequence with signature

$$E^2_{p,q} = H_p(B\mathrm{U}_1\times BS_4, \Omega_q^{\mathrm{SO}}(\mathrm{pt})) \Longrightarrow \Omega_{p+q}^{\mathrm{SO}}(B\mathrm{U}_1\times BS_4). \tag{4.5.77}$$

The coefficients are sums of $\mathbb{Z}$ and $\mathbb{Z}/2$; since we only care about 3-torsion, we can ignore the $\mathbb{Z}/2$ summands, whose differentials cannot map nontrivially to or from any 3-torsion element. The only 3-torsion on the E^2-page in total degree less than 7 is a single $\mathbb{Z}/3$ summand in each of $E^2_{3,0}$ and $E^2_{5,0}$, coming from our calculation above of 3-torsion in homology. These 3-torsion summands cannot participate in any nonzero differentials: they do not map to each other, and cannot receive any differentials from free summands, or from the 7-line (which we have not calculated). Thus they persist to the E^∞-page. It is a priori possible more 3-torsion is created from free summands on the E^2-page, which could happen if a differential maps from a free summand to another free summand. All free summands are in even total degree, though, so this does not happen, and the only 3-torsion in $\Omega_k^{\mathrm{SO}}(B\mathrm{U}_1 \times BS_4)$, for $k < 7$, is two $\mathbb{Z}/3$ summands in degrees 3 and 5. $\square$

Next, we compute the 2-torsion using the Adams spectral sequence over $\mathcal{E}(1)$.

For the spinless case, recall from (4.5.68) (drawn in Figure 31) the calculation of $\widetilde{H}^*((BS_4)^{3-V_\lambda};\mathbb{Z}/2)$ as an $\mathcal{A}(1)$-module. There are isomorphisms of $\mathcal{E}(1)$-modules $N_4 \cong \mathcal{\mathring{O}} \oplus \Sigma\mathcal{E}(1) \oplus \Sigma^3\mathbb{Z}/2$ and $\mathcal{A}(1) \cong \mathcal{E}(1) \oplus \Sigma^2\mathcal{E}(1)$, so as $\mathcal{E}(1)$-modules,

$$\widetilde{H}^*((BS_4)^{3-V_\lambda};\mathbb{Z}/2) \cong \mathcal{\mathring{O}} \oplus \Sigma\mathcal{E}(1) \oplus \Sigma^3\mathbb{Z}/2 \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^4\mathcal{\mathring{O}} \oplus \Sigma^5\mathcal{E}(1) \oplus \Sigma^5\mathcal{E}(1) \oplus \Sigma^5\mathcal{E}(1) \oplus P, \tag{4.5.78}$$

where P is 6-connected. We draw this in Figure 34, left.

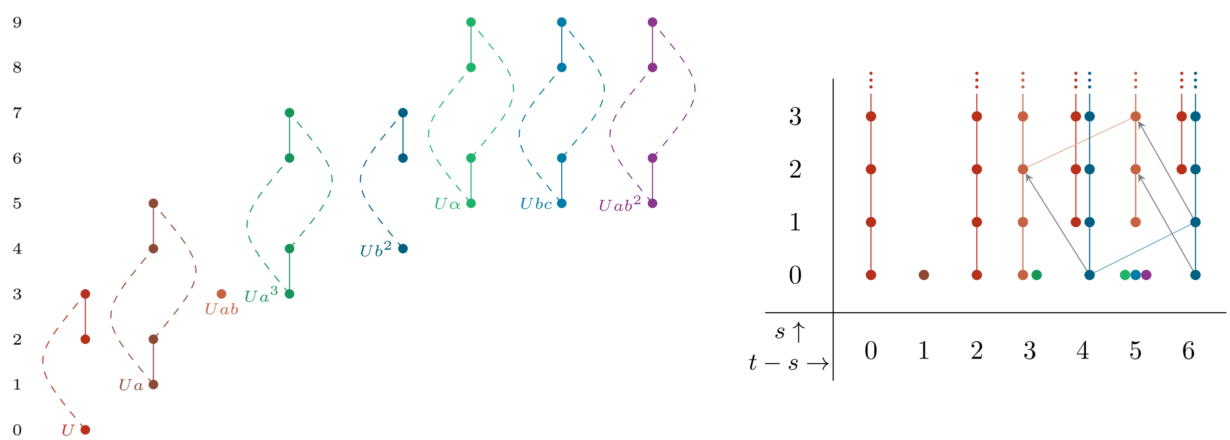


FIGURE 34. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*((BS_4)^{3-V_\lambda};\mathbb{Z}/2)$ in low degrees. The pictured submodule contains all elements of degrees at most 6. Here $\alpha := a^5 + a^3b$. Right: the corresponding Ext, which is the E_2-page of the Adams spectral sequence for $\widetilde{ku}_*((BS_4)^{3-V})$. Some nonzero v_1-actions are hidden for clarity.

To draw the E_2-page of the Adams spectral sequence, use the computations of $\mathrm{Ext}(\mathcal{\mathring{O}})$ from (4.4.56) and $\mathrm{Ext}_{\mathcal{E}(1)}(\mathbb{Z}/2)$ from (1.1.45) to obtain Figure 34, right. For clarity, we do not draw most v_1-actions. There may be differentials in this range, though we do not determine whether they are the d_2s pictured.

From Figure 34, right, $\widetilde{ku}_0((BS_4)^{3-V_\lambda}) \cong \mathbb{Z}$, so Lemma 4.3.16 implies there is a single free summand in each even degree and the odd-degree ku-groups are torsion. Therefore, one of the towers on the 4-line must admit a nontrivial d_r differential to the tower on the 3-line, and in fact, v_1-equivariance of the differentials implies that tower on the 4-line must be the blue one coming from $\Sigma^4\mathcal{\mathring{O}}$. The remaining tower must survive, so on the E_∞-page, the 3-line has its $\mathbb{Z}/2$ summand and a $\mathbb{Z}/2^r$ summand coming from the red tower, and the 4-line has a single $\mathbb{Z}$ summand left. The results on $\widetilde{ku}_5$ and $\widetilde{ku}_6$ follow from v_1- and h_0-equivariance of d_r.

On to the spin-1/2 case. We factor $ku_*(BS_4) \cong ku_*(\mathrm{pt}) \oplus \widetilde{ku}_*(BS_4)$. In the proof of Theorem 4.5.70, we split $\Sigma^\infty BS_4 \simeq \Sigma^\infty B\mathbb{Z}/2 \vee M$ and determined the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$. Combining this with Nguyen's computation [**Ngu09**, Theorem 2.3.1] of the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(BS_4;\mathbb{Z}/2)$, we have that as $\mathcal{E}(1)$-modules,

$$\widetilde{H}^*(M;\mathbb{Z}/2) \cong \Sigma^2\mathcal{E}(1) \oplus \Sigma^3\mathcal{\mathring{O}} \oplus \Sigma^4\mathbb{Z}/2 \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^6\mathcal{E}(1) \oplus \Sigma^6\mathcal{E}(1) \oplus P, \tag{4.5.79}$$

where P is 6-connected. We draw this in Figure 35, left.

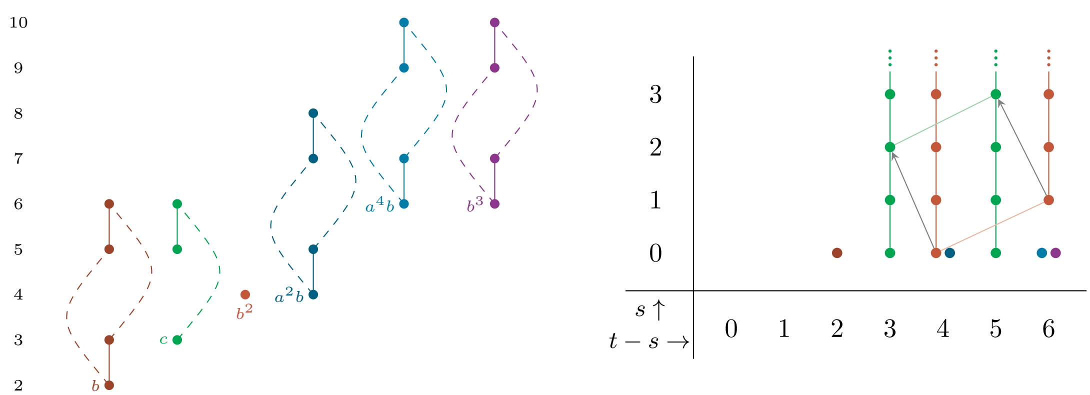


Figure 35. Left: the $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ in low degrees. The pictured submodule contains all elements of degrees at most 6. Right: the corresponding Ext, which is the E_2-page of the Adams spectral sequence computing $\widetilde{ku}_*(M)$. Some v_1-actions are hidden to declutter the diagram.

For each of these modules N (except P, which as usual is too high-degree to be relevant), we already calculated $\mathrm{Ext}^{s,t}_{\mathcal{E}(1)}(N,\mathbb{Z}/2)$: for $\mathcal{\mathring{O}}$, see (4.4.56), and for $\mathbb{Z}/2$, see (1.1.45). Therefore the E_2-page for the Adams spectral sequence is as drawn in Figure 35, right. Most of the v_1-actions are hidden to make the diagram clearer. We indicate locations of some possible differentials, but they are not necessarily d_2s.

Lemma 4.3.16 implies $\widetilde{ku}_*(BS_4)$ is torsion, so all towers present on the E_2-page must emit or receive differentials. Thus there is some $r' \geq 2$ such that the green tower on the 3-line is killed by a $d_{r'}$ emerging from the orange tower on the 4-line; therefore on the E_∞-page, the 4-line contains only the $\mathbb{Z}/2$ summand in

$E_\infty^{0,4}$, and the 3-line contains r' $\mathbb{Z}/2$ summands, the remains of the tower. For $\widetilde{ku}_5$ and $\widetilde{ku}_6$, v_1-equivariance of $d_{r'}$ determines the E_∞-page in the same way.

It remains to add in the summands corresponding to $ku_*(\mathrm{pt})$ and $\widetilde{ku}_*(B\mathbb{Z}/2)$; the former contributes a $\mathbb{Z}$ summand in each even dimension, and the latter contributes $\mathbb{Z}/2$ in degree 1, $\mathbb{Z}/4$ in degree 3, and $\mathbb{Z}/8$ in degree 5, by work of Hashimoto [**Has83**, Theorem 3.1]. □

4.5.5. Full octahedral symmetry. The full group of symmetries of the octahedron, including orientation-reversing ones, is isomorphic to $G := A_4 \times \mathbb{Z}/2$. Let $\lambda\colon G \to \mathrm{O}_3$ denote the corresponding three-dimensional real representation of G, and $V_\lambda \to BG$ denote the associated vector bundle. We saw in §4.5.4 the pullback of V_λ along $BS_4 \to BG$ is not pin^c, so V_λ is also not pin^c, and therefore is also not pin$^-$.

The Künneth formula and Proposition 4.5.32 together imply

$$H^*(BG;\mathbb{Z}/2) \cong \mathbb{Z}/2[x,a,b,c]/(ac), \tag{4.5.80}$$

where $|x| = |a| = 1$, $|b| = 2$, and $|c| = 3$.

Lemma 4.5.81. *$w_1(V_\lambda) = x$ and $w_2(V_\lambda) = b + x^2$.*

Proof. For w_1, we know $w_1(V_\lambda) \neq 0$ because V_λ is unorientable, but because $V_\lambda|_{BS_4}$ is orientable, $w_1(V_\lambda)$ cannot be a or $x + a$, leaving $w_1(V_\lambda) = x$.

For w_2, we know the pullback of V_λ to BS_4 has $w_2(V|_{S_4}) = b$. If $i\colon B\mathbb{Z}/2 \to BG$ is induced by the inclusion of a reflection in G, then $i^*\lambda$ decomposes as a direct sum of three copies of the sign representation, so $i^*V_\lambda \cong 3\sigma$. Therefore $i^*w_2(V_\lambda) = x^2$, uniquely constraining $w_2(V_\lambda) = b + x^2$. □

4.5.5.1. *Class D, spinless case.* The FCEP says we should study the spin bordism of $(BG)^{3-V_\lambda}$. We will argue as we did in the case of pyritohedral symmetry in §4.5.2, replacing $3 - V_\lambda$ with a virtual vector bundle whose Adams E_2-page is isomorphic to that of $(BG)^{3-V_\lambda}$, but which is easier to calculate. This isomorphism did not come from a map of spectra, so cannot tell us anything about differentials or hidden extensions, but just as for pyritohedral symmetry, we will see that for entirely formal reasons, all differentials vanish and all hidden extensions split in the range we need. Using the twisted Künneth formula, $\widetilde{H}^*((BG)^{3-V_\lambda})$ contains no odd-primary torsion, so neither does $\widetilde{\Omega}_*^{\mathrm{Spin}}((BG)^{3-V_\lambda})$, so using the 2-primary Adams spectral sequence suffices.

For the rest of this section, all homology and cohomology is understood to be with $\mathbb{Z}/2$ coefficients.

Lemma 4.5.82. *Let $E \to BG$ denote the virtual vector bundle induced from the virtual representation*

$$2 - (V_\lambda|_{S_4} \boxplus (-\sigma)). \tag{4.5.83}$$

Then, there is an isomorphism of $\mathcal{A}(1)$-modules $\widetilde{H}^*((BG)^{3-V_\lambda}) \cong \widetilde{H}^*((BG)^E)$, hence an isomorphism between the E_2-pages of the Adams spectral sequences for $ko \wedge (BG)^{3-V_\lambda}$ and $ko \wedge (BG)^E$.

Proof. The E_2-pages of these Adams spectral sequences are determined by the $\mathcal{A}(1)$-module structures on cohomology, which are in turn determined by w_1 and w_2 of the virtual bundles $3 - V_\lambda$ and E. Since $w_1(E) = x$ and $w_2(E) = u$, then for $i = 1, 2$, $w_i(3 - V_\lambda) = w_i(E)$. ☐

Because E is induced from a representation which is an exterior sum, its Thom spectrum splits as

$$(BG)^E \simeq (BS_4)^{3-V_\lambda|_{S_4}} \wedge (B\mathbb{Z}/2)^{\sigma-1} \tag{4.5.84}$$

The Künneth theorem then simplifies the E_2-page:

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(\widetilde{H}^*((BS_4)^{3-V_\lambda|_{S_4}}) \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}), \mathbb{Z}/2). \tag{4.5.85}$$

Campbell [**Cam17**, Figure 6.1] computes the $\mathcal{A}(1)$-module structure on $\widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1})$, and we computed $\widetilde{H}^*((BS_4)^{3-V_\lambda})$ in (4.5.68) (drawn in Figure 31).

Proposition 4.5.86. *There is an isomorphism of $\mathcal{A}(1)$-modules*

$$\widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}) \otimes_{\mathbb{Z}/2} N_4 \cong \Sigma N_5 \oplus (V_2 \otimes_{\mathbb{Z}/2} \mathcal{A}(1)) \oplus P_2, \tag{4.5.87}$$

where N_5 is as in Figure 36, V_2 is a graded vector space with a homogeneous basis in degrees $\{0, 2, 3, 4\}$, and P_2 is 4-connected.

Proof. Compute directly, by hand or by computer. ☐

Recall from (4.5.68) (drawn in Figure 31) the $\mathcal{A}(1)$-module structure on $(BS_4)^{3-V_\lambda}$. Margolis' theorem (Theorem 4.3.14) splits off a $\Sigma^k H\mathbb{Z}/2$ summand from $ko \wedge (BS_4)^{3-V_\lambda}$ for every direct summand of $\Sigma^k \mathcal{A}(1)$ in $\widetilde{H}^*((BS_4)^{3-V_\lambda})$; below degree 8, this occurs for $k = 3, 5$. Therefore, by the same line of reasoning as in §4.5.2, there is a spectrum Y' such that

$$\widetilde{ko}_n((BG)^{3-V_\lambda}) \cong \pi_n(Y') \oplus \widetilde{H}_{n-3}((B\mathbb{Z}/2)^{\sigma-1}) \oplus \widetilde{H}_{n-5}((B\mathbb{Z}/2)^{\sigma-1}), \tag{4.5.88}$$

and as $\mathcal{A}$-modules,

$$\widetilde{H}^*(Y') \cong \mathcal{A} \otimes_{\mathcal{A}(1)} (N_4 \oplus \Sigma^4 N_4 \oplus P_3) \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}), \tag{4.5.89}$$

where P_3 is a 4-connected $\mathcal{A}(1)$-module. Therefore the change-of-rings formula (1.1.43) applies to the E_2-page of the Adams spectral sequence for $\pi_*(Y')$, showing

$$E_2^{s,t}(Y') \cong \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}((N_4 \oplus \Sigma^4 N_4 \oplus P_3) \otimes_{\mathbb{Z}/2} \widetilde{H}^*((B\mathbb{Z}/2)^{\sigma-1}), \mathbb{Z}/2). \tag{4.5.90}$$

To calculate the spin bordism groups of $(BG)^{3-V_\lambda}$, we will work with this spectral sequence, adding the summands corresponding to $\Sigma^3 H\mathbb{Z}/2$ and $\Sigma^5 H\mathbb{Z}/2$ at the end.

Theorem 4.5.91. *The first few spin bordism groups of $(BG)^{3-V_\lambda}$ are*

$$\begin{aligned}
\widetilde{\Omega}_0^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_1^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong \mathbb{Z}/2\\
\widetilde{\Omega}_2^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_3^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong (\mathbb{Z}/2)^{\oplus 2}\\
\widetilde{\Omega}_4^{\mathrm{Spin}}((BG)^{3-V_\lambda}) &\cong (\mathbb{Z}/2)^{\oplus 4},
\end{aligned}$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}}((BG)^{3-V_\lambda})$ is torsion, so the 0^{th} $(S_4 \times \mathbb{Z}/2)$-equivariant phase homology group for this case is isomorphic to $(\mathbb{Z}/2)^{\oplus 4}$.

Proof. Proposition 4.5.86 and (4.5.90) together imply the E_2-page for Y' is

$$E_2^{s,t}(Y') \cong \mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(\Sigma N_5 \oplus V_2 \otimes_{\mathbb{Z}/2} \mathcal{A}(1) \oplus \Sigma^5 N_5 \oplus \Sigma^4 V_2 \otimes_{\mathbb{Z}/2} \mathcal{A}(1) \oplus P, \mathbb{Z}/2), \tag{4.5.92}$$

where P is 4-connected. We will see that the E_2-page in $t-s \le 4$ is empty for $s \ge 2$, so there can be no differentials involving $\mathrm{Ext}(P)$ in the range we care about.

Our first order of business is therefore to determine $\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(N_5, \mathbb{Z}/2)$ for small s, t. There is an extension of $\mathcal{A}(1)$-modules

$$0 \longrightarrow R_3 \longrightarrow N_5 \longrightarrow \Sigma R_0 \longrightarrow 0, \tag{4.5.93}$$

which we draw in Figure 36, left, fitting $\mathrm{Ext}_{\mathcal{A}(1)}^{s,t}(N_5, \mathbb{Z}/2)$ into a long exact sequence (Figure 36, right). The $\mathcal{A}(1)$-module R_3 and its Ext are calculated in the range we need by Freed-Hopkins [**FH16a**, Figure 5, case

$s = 3$] and Beaudry-Campbell [**BC18**, Figures 32, 33]. In the range pictured, there are two boundary maps in Figure 36, right, which could be nonzero; the existence of a nonzero map $N_5 \to \Sigma^4\mathbb{Z}/2$ forces the boundary map $\delta\colon \mathrm{Ext}^{0,4}(R_3) \to \mathrm{Ext}^{1,4}(\Sigma R_0)$ to vanish. We do not need to know whether the other pictured boundary map vanishes.

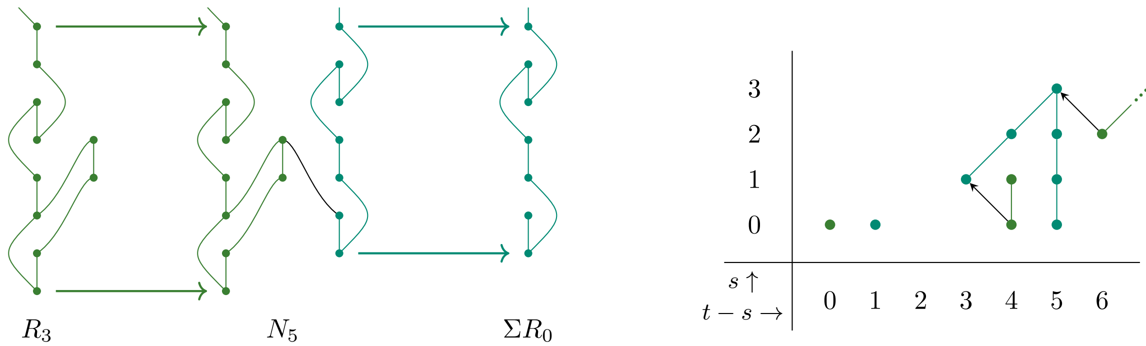


FIGURE 36. Left: the $\mathcal{A}(1)$-module N_5 in the extension (4.5.93). Right: the corresponding long exact sequence in Ext.

Hence Figure 37 shows the E_2-page for computing $\pi_*(Y')$.

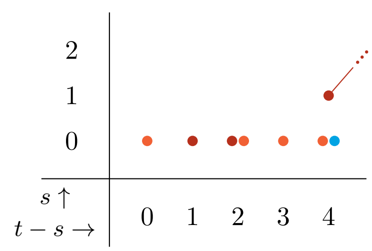


FIGURE 37. The E_2-page for computing $\pi_*(Y')$. See the proof of Theorem 4.5.91 for more details.

The 4-line is concentrated in filtration 0 and 1, and so there can be neither nonzero differentials nor nontrivial extension problems involving elements of degree 4 or less. This accounts for $\pi_*(Y')$; for the each factor of $\widetilde{H}_{*-\ell}((B\mathbb{Z}/2)^{\sigma-1})$, add a single $\mathbb{Z}/2$ summand in degrees ℓ and above. $\square$

4.5.5.2. *Class D, spin-1/2 case.* Now we ask for the symmetries to mix. Let $f^D_{1/2}$ denote the local system of symmetry types for this case. By Theorem 4.2.11, we consider the spin bordism of $X := (BS_4 \times B\mathbb{Z}/2)^{\mathrm{Det}(V_\lambda)-1}$, because V_λ is not pin$^-$. The isomorphism $\mathrm{Det}\, V_\lambda \cong 0 \boxplus \sigma$ provides an isomorphism $X \simeq (BS_4)_+ \wedge (B\mathbb{Z}/2)^{\sigma-1}$, so (4.2.10b) implies the spin bordism of this spectrum computes the pin$^-$ bordism of BS_4, which could be independently interesting.

Theorem 4.5.94. *The first few spin bordism groups of X are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}}(X) &\cong \mathbb Z/2\\
\widetilde\Omega_1^{\mathrm{Spin}}(X) &\cong (\mathbb Z/2)^{\oplus 2}\\
\widetilde\Omega_2^{\mathrm{Spin}}(X) &\cong \mathbb Z/8\oplus\mathbb Z/4\oplus\mathbb Z/2\\
\widetilde\Omega_3^{\mathrm{Spin}}(X) &\cong (\mathbb Z/2)^{\oplus 4}\\
\widetilde\Omega_4^{\mathrm{Spin}}(X) &\cong (\mathbb Z/2)^{\oplus 2}.
\end{aligned}$$

Since $\widetilde\Omega_5^{\mathrm{Spin}}(X)$ is torsion by Lemma 4.3.16, $Ph_0^{S_4\times\mathbb Z/2}(\mathbb R^3; f^D_{1/2})\cong\mathbb Z/2$.

Proof. As usual, Lemma 4.3.22 spits X as a sum of $(B\mathbb Z/2)^{\sigma-1}$ and another spectrum M, where $\widetilde H^*(M;\mathbb Z/2)$ is complementary in $\widetilde H^*(X;\mathbb Z/2)$ to the space spanned by $\{Uw_1(\lambda)^k\}$. The $(B\mathbb Z/2)^{\sigma-1}$ summand gives us pin$^-$ bordism, and we focus on M.

We have $w_1(\mathrm{Det}(V_\lambda)-1)=w_1(V_\lambda)=x$ and $w_2(\mathrm{Det}\,V_\lambda-1)=0$; this and the $\mathcal A$-module structure on $BS_4\times B\mathbb Z/2$ calculated in (4.5.80) determine the $\mathcal A(1)$-module structure on M. We obtain an isomorphism of $\mathcal A(1)$-modules

$$\widetilde H^*(M;\mathbb Z/2)\cong \Sigma R_5\oplus\Sigma^2R_3\oplus\Sigma^3\mathcal A(1)\oplus\Sigma^3\mathcal A(1)\oplus\Sigma^3\mathcal A(1)\oplus\Sigma^3\mathcal A(1)\oplus\Sigma^4\mathcal A(1)\oplus\Sigma^4\mathcal A(1)\oplus P, \tag{4.5.95}$$

where P is 4-connected. We will see momentarily that for $t-s\le 4$, $E_2^{s,t}$ is empty for $s\ge 2$; this and the 4-connectedness of P imply its contribution to the E_2-page cannot affect the spectral sequence in degrees $t-s\le 4$, which is all we need. We draw these summands, except for P, in Figure 38.

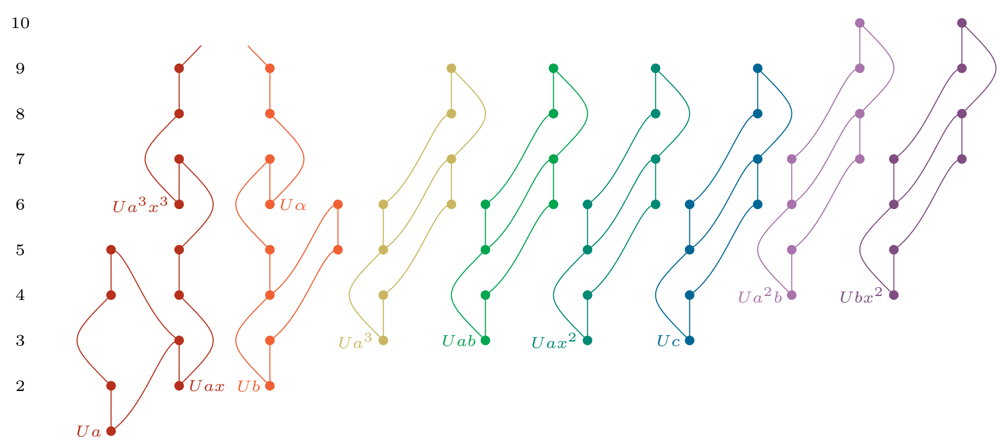


Figure 38. The $\mathcal A(1)$-module structure on $\widetilde H^*(M;\mathbb Z/2)$ in low degrees. The pictured summand contains all classes in degrees 4 and below. Here $\alpha := b^2x^2+a^2b^2+c^2$.

Freed-Hopkins [**FH16a**, Figure 5, cases $s = \pm 3$] and Beaudry-Campbell [**BC18**, Figures 32, 33, 37] calculate $\mathrm{Ext}(R_5)$ and $\mathrm{Ext}(R_3)$ in the degrees we need, and we draw the E_2-page of the Adams spectral sequence in Figure 39. This collapses, and it and the pin$^-$ bordism groups from the $(B\mathbb{Z}/2)^{\sigma-1}$ summand,

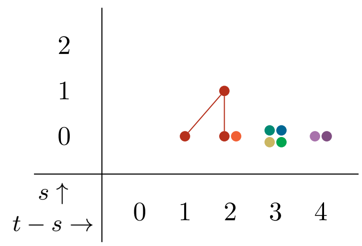


FIGURE 39. The E_2-page for computing $\widetilde{ko}_*(M)$. See the proof of Theorem 4.5.94 for more information.

which are computed in [**ABP69, KT90b**], sum together to the groups in the theorem. $\square$

4.5.5.3. *Class A, spinless case.* Let f_0^A denote the local system of symmetry types for this case. We want to calculate $\widetilde{\Omega}_*^{\mathrm{Spin}^c}((BG)^{3-V_\lambda})$. Using the twisted Künneth formula, $\widetilde{H}^*((BG)^{3-V_\lambda};\mathbb{Z}/2)$ is 2-torsion, and therefore $\widetilde{\Omega}_*^{\mathrm{Spin}^c}((BG)^{3-V_\lambda})$ is too, so it suffices to use the 2-primary Adams spectral sequence.

Theorem 4.5.96. *The first few spin^c bordism groups of $(BG)^{3-V_\lambda}$ are:*

$$\widetilde{\Omega}_0^{\mathrm{Spin}^c}((BG)^{3-V_\lambda}) \cong \mathbb{Z}/2$$

$$\widetilde{\Omega}_1^{\mathrm{Spin}^c}((BG)^{3-V_\lambda}) \cong \mathbb{Z}/2$$

$$\widetilde{\Omega}_2^{\mathrm{Spin}^c}((BG)^{3-V_\lambda}) \cong (\mathbb{Z}/2)^{\oplus 3}$$

$$\widetilde{\Omega}_3^{\mathrm{Spin}^c}((BG)^{3-V_\lambda}) \cong (\mathbb{Z}/2)^{\oplus 3}$$

$$\widetilde{\Omega}_4^{\mathrm{Spin}^c}((BG)^{3-V_\lambda}) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 4},$$

and $\widetilde{\Omega}_5^{\mathrm{Spin}^c}((BG)^{3-V_\lambda})$ is torsion. Hence $Ph_0^{S_4\times\mathbb{Z}/2}(\mathbb{R}^3; f_0^A) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 4}$.

PROOF. There is an isomorphism of $\mathcal{E}(1)$-modules

$$N_5 \cong \mathcal{E}(1) \oplus \Sigma R_0 \oplus \Sigma^2 R_0, \tag{4.5.97}$$

hence another isomorphism of $\mathcal{E}(1)$-modules

$$\widetilde{H}^*((BG)^{3-V_\lambda}) \cong (V_c \otimes_{\mathbb{Z}/2} \mathcal{A}(1)) \oplus \Sigma^2 R_0 \oplus \Sigma^3 R_0 \oplus P_c, \tag{4.5.98}$$

where P_c is 4-connected and V_c is a graded vector space with a homogeneous basis of elements in degrees $\{0,1,2,2,3,3,4,4,4,4\}$. Therefore we can draw the E_2-page of the Adams spectral sequence, and do so in Figure 40.

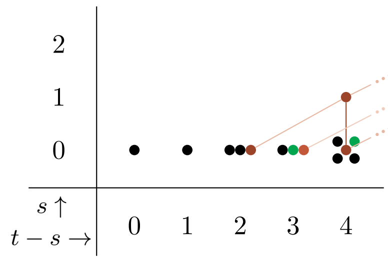


FIGURE 40. The E_2-page for computing $\widetilde{ku}_*(M)$.

Below degree 5, there are no nonzero differentials, because there is nothing in Adams filtration 2 or higher. And degree considerations rule out hidden extensions, so we are done. □

4.5.5.4. *Class A, spin-1/2 case.* Because V_λ is not pin^c, Theorem 4.2.24 tells us to compute the spin^c bordism groups of $X \coloneqq (BS_4 \times B\mathbb{Z}/2)^{\mathrm{Det}(V_\lambda)-1}$.

Theorem 4.5.99. *The first few spin^c bordism groups of X are*

$$\widetilde\Omega_0^{\mathrm{Spin}^c}(X) \cong \mathbb{Z}/2$$

$$\widetilde\Omega_1^{\mathrm{Spin}^c}(X) \cong \mathbb{Z}/2$$

$$\widetilde\Omega_2^{\mathrm{Spin}^c}(X) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 2}$$

$$\widetilde\Omega_3^{\mathrm{Spin}^c}(X) \cong (\mathbb{Z}/2)^{\oplus 4}$$

$$\widetilde\Omega_4^{\mathrm{Spin}^c}(X) \cong \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 4}.$$

As $\widetilde\Omega_5^{\mathrm{Spin}^c}(X)$ is torsion, the 0^{th} $(S_4 \times \mathbb{Z}/2)$-equivariant phase homology group for this case is isomorphic to $\mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 4}$.

PROOF. By Lemma 4.3.22, X splits as $(B\mathbb{Z}/2)^{\sigma-1} \vee M$, where $\widetilde H^*(M;\mathbb{Z}/2)$ is isomorphic to a complementary subspace to the subspace $\mathbb{Z}/2 \cdot \{Ux^k\}$ inside $\widetilde H^*(X;\mathbb{Z}/2)$. As usual, the $(B\mathbb{Z}/2)^{\sigma-1}$ summand contributes pin^c bordism groups to the final answer, so we focus on M. The $\mathcal{A}(1)$-module structure we computed in (4.5.95) and drew in Figure 38 tells us the $\mathcal{E}(1)$-structure; here, we use that $R_5 \cong \mathcal{E}(1) \oplus \Sigma R_0$ and $R_3 \cong \mathcal{E}(1) \oplus \Sigma^2 R_0$ as $\mathcal{E}(1)$-modules. Therefore, there is an $\mathcal{E}(1)$-module isomorphism

$$\widetilde H^*(M;\mathbb{Z}/2) \cong \Sigma\mathcal{E}(1) \oplus \Sigma^2 R_0 \oplus \Sigma^2\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^3\mathcal{E}(1) \oplus \Sigma^4 R_0 \oplus \Sigma^4\mathcal{E}(1) \oplus \Sigma^4\mathcal{E}(1) \oplus P, \tag{4.5.100}$$

where P is 4-connected. Therefore to infer anything about $\widetilde{\Omega}_4^{\mathrm{Spin}^c}(M)$ from this spectral sequence, we must argue that P does not affect it; this will follow when we see the $t-s=4$ line of the E_2-page is empty in Adams filtration 2 and above, so there can be no nonzero differentials from the 5-line to the 4-line. We draw (4.5.100) in Figure 41.

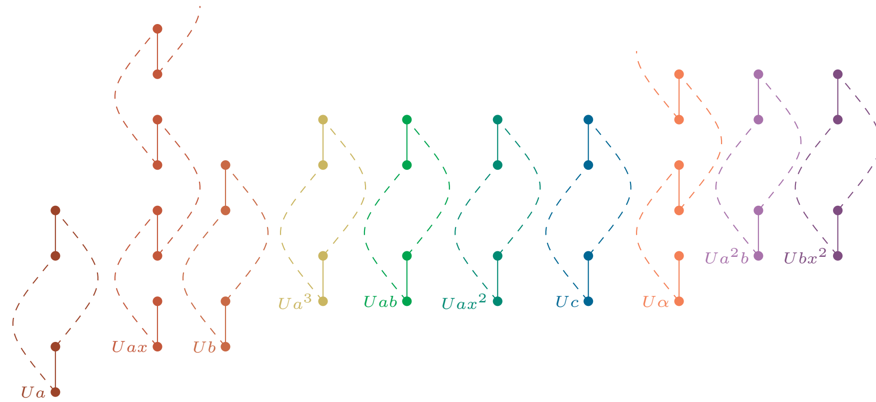


FIGURE 41. The $\mathcal{E}(1)$-module structure on $\widetilde{H}^*(M;\mathbb{Z}/2)$ in low degrees. Here $\alpha \coloneqq abx + b^2 + cx$. This submodule contains all elements in degrees 4 and below.

Recalling $\mathrm{Ext}(R_0)$ from Proposition 4.4.49, the E_2-page of the Adams spectral sequence for $\widetilde{ku}_*(M)$ is drawn in Figure 42. In this range, the spectral sequence collapses, so we read off $\widetilde{\Omega}_*^{\mathrm{Spin}^c}(M)$ and combine it

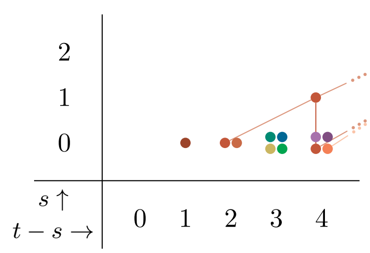


FIGURE 42. The E_2-page for computing $\widetilde{ku}_*(M)$.

with pin^c bordism as computed in [**BG87a, BG87b**] to conclude. ☐

4.5.6. Chiral icosahedral symmetry. Let $\lambda\colon A_5 \to \mathrm{SO}_3$ denote the representation given by chiral icosahedral symmetry, and as usual let $V_\lambda \to BA_5$ denote the associated vector bundle.

Remark 4.5.101. Unlike the previous symmetry groups we studied, icosahedral symmetry is incompatible with translations, and there are no space groups whose underlying point group is either the chiral icosahedral group or the full icosahedral group. This means one should not expect to realize any phases equivariant for these symmetry groups as a lattice Hamiltonian system on a periodic lattice on $\mathbb{R}^3$. This does not rule out the possibility of interesting phases with an icosahedral symmetry: there are examples of phases studied via lattice Hamiltonian realizations on lattices in great generality, such as the toric code model in [**Fre19**, §2.3] and Example 1.4.7; the GDS model in [**FH16b, FHHT20**], and Chapter 2 of this thesis; and the phases on aperiodic lattices studied by Huang-Wu-Liu [**HWL20**]. In a similar vein, it may be possible for a Hamiltonian on an aperiodic lattice with icosahedral symmetry to model a nontrivial crystalline SPT. See [**VLP⁺19**] for an example of how such an implementation might look.

For icosahedral symmetry, the hard work is behind us. Let $\lambda\colon A_5 \to \mathrm{O}_3$ denote the representation as the orientation-preserving symmetries of the icosahedron. The restriction to $A_4 \subset A_5$ corresponds to symmetries that preserve a concentric tetrahedron. Let $V_\lambda \to BA_5$ be the associated bundle to λ.

Lemma 4.5.102. *The inclusion $\varphi\colon A_4 \hookrightarrow A_5$ induces an equivalence on mod 2 cohomology. Hence φ induces 2-primary equivalences $\Sigma^\infty(BA_4)_+ \to \Sigma^\infty(BA_5)_+$ and $(BA_4)^{3-\varphi^*(V_\lambda)} \to (BA_5)^{3-V_\lambda}$.*

Proof. The first part is Lemma 4.3.19: here $[A_5 : A_4] = 5$, $P = \mathbb{Z}/2 \times \mathbb{Z}/2$, and for both A_4 and A_5, $N(P)/P \cong \mathbb{Z}/3$.

For the second part, the Thom isomorphism theorem tells us $\varphi'\colon (BA_4)^{3-\varphi^*(V_\lambda)} \to (BA_5)^{3-V_\lambda}$ induces an isomorphism on mod 2 cohomology. The desired 2-primary equivalences then follow from the mod 2 Whitehead theorem [**Ser53**, Chapitre III, Théorème 3]. ☐

We can therefore reuse the calculations we made at the prime 2 in §4.5.1 to obtain the 2-primary parts of $\widetilde\Omega_k^{\mathrm{Spin}}((BA_5)^{3-V_\lambda})$ and $\Omega_k^{\mathrm{Spin}}(BA_5)$; the odd-primary pieces are different, but not hard.

Proposition 4.5.103. *The only odd-primary torsion in $H_k(BA_5)$ for $k < 7$ is contained in $H_3(BA_5) \cong \mathbb{Z}/30$.*

Proof sketch. One can compute this using Gap; we also indicate how to do it by hand. Since $|A_5| = 60 = 2^2 \cdot 3 \cdot 5$, there is no p-primary torsion for $p > 5$, so it suffices to determine $H^k(BA_5; \mathbb{Z}/3)$ and $H^k(BA_5; \mathbb{Z}/5)$ in low degrees. This can be done using the theorem of Adem-Milgram [**AM04**, Theorem II.6.8] mentioned above, since the Sylow 3- and 5-subgroups of A_5 are abelian. ☐

Corollary 4.5.104. *In $\widetilde\Omega_k^{\mathrm{Spin}}((BA_5)^{3-V_\lambda})$ and $\Omega_k^{\mathrm{Spin}}(BA_5)$, the only odd-primary torsion for $k < 7$ is a $\mathbb{Z}/15$ in degree 3.*

Proof. As usual, we use the fact that $\Omega_*^{\mathrm{Spin}} \to \Omega_*^{\mathrm{SO}}$ is an isomorphism on odd-primary torsion, together with the Thom isomorphism $\widetilde\Omega_*^{\mathrm{SO}}((BA_5)^{3-V_\lambda}) \cong \Omega_*^{\mathrm{SO}}(BA_5)$, to reduce to showing the claim for $\Omega_k^{\mathrm{SO}}(BA_5)$. For this, use the Atiyah-Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(BA_5; \Omega_q^{\mathrm{SO}}(\mathrm{pt})) \Longrightarrow \Omega_{p+q}^{\mathrm{SO}}(BA_5). \tag{4.5.105}$$

On the E^2-page, the only odd-primary torsion in total degree below 7 is $\mathbb{Z}/15 \subset E^2_{3,0} = H_3(BA_5)$. In all differentials involving $E^r_{3,0}$, the other group is zero, so this odd-primary torsion lives to the E^∞-page.

We also must check that the free summands in total degree below 7 do not receive differentials that produce more odd-primary torsion. There are only two such free summands, in $E^2_{0,0}$ and $E^2_{0,4}$, and they can only receive differentials from 2-torsion abelian groups, so that does not happen. $\square$

Now we need to combine this with the 2-primary summands. For $(BA_5)^{3-V_\lambda}$, we need $\Omega_*^{\mathrm{Spin}}((BA_4)^{3-\varphi^* V_\lambda})$, which we computed in Theorem 4.5.4. For BA_5, we need $\Omega_*^{\mathrm{Spin}}(BA_4)$; in the degrees we need, this is isomorphic to $ko_*(BA_4)$, which Bruner-Greenlees compute in [**BG10**, §7.7.E].

Theorem 4.5.106. *The low-degree spin bordism groups of $(BA_5)^{3-V}$ and BA_5 are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong \mathbb{Z} & \Omega_0^{\mathrm{Spin}}(BA_5) &\cong \mathbb{Z}\\
\widetilde\Omega_1^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong 0 & \Omega_1^{\mathrm{Spin}}(BA_5) &\cong \mathbb{Z}/2\\
\widetilde\Omega_2^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong 0 & \Omega_2^{\mathrm{Spin}}(BA_5) &\cong \mathbb{Z}/2\oplus\mathbb{Z}/2\\
\widetilde\Omega_3^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong \mathbb{Z}/30 & \Omega_3^{\mathrm{Spin}}(BA_5) &\cong \mathbb{Z}/60\\
\widetilde\Omega_4^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong \mathbb{Z} & \Omega_4^{\mathrm{Spin}}(BA_5) &\cong \mathbb{Z}\\
\widetilde\Omega_5^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong \mathbb{Z}/2\oplus\mathbb{Z}/2 & \Omega_5^{\mathrm{Spin}}(BA_5) &\cong 0\\
\widetilde\Omega_6^{\mathrm{Spin}}((BA_5)^{3-V_\lambda}) &\cong \mathbb{Z}/2 & \Omega_6^{\mathrm{Spin}}(BA_5) &\cong \mathbb{Z}/2.
\end{aligned}$$

Hence the 0^{th} A_5-equivariant phase homology groups vanish for both spinless and spin-$1/2$ fermions.

Finally, class A. Since V_λ is not pin^c, because its restriction to A_4 is not (Lemma 4.5.7), we care about $(BA_5)^{\mathrm{Det}(V_\lambda)-1} \cong (BA_5)_+$ in the spin-1/2 case, because V_λ is orientable. Let f_0^A, resp. $f_{1/2}^A$, denote the equivariant local systems of symmetry types for the class A spinless, resp. spin-1/2 cases.

Theorem 4.5.107. *The low-degree spin^c bordism groups of $(BA_5)^{3-V_\lambda}$ and BA_5 are*

$$\begin{aligned}
\widetilde\Omega_0^{\mathrm{Spin}^c}((BA_5)^{3-V_\lambda}) &\cong \mathbb{Z} & \Omega_0^{\mathrm{Spin}^c}(BA_5) &\cong \mathbb{Z}\\
\widetilde\Omega_1^{\mathrm{Spin}^c}((BA_5)^{3-V_\lambda}) &\cong 0 & \Omega_1^{\mathrm{Spin}^c}(BA_5) &\cong 0\\
\widetilde\Omega_2^{\mathrm{Spin}^c}((BA_5)^{3-V_\lambda}) &\cong \mathbb{Z} & \Omega_2^{\mathrm{Spin}^c}(BA_5) &\cong \mathbb{Z}\oplus\mathbb{Z}/2\\
\widetilde\Omega_3^{\mathrm{Spin}^c}((BA_5)^{3-V_\lambda}) &\cong \mathbb{Z}/30 & \Omega_3^{\mathrm{Spin}^c}(BA_5) &\cong \mathbb{Z}/30\\
\widetilde\Omega_4^{\mathrm{Spin}^c}((BA_5)^{3-V_\lambda}) &\cong \mathbb{Z}^2 & \Omega_4^{\mathrm{Spin}^c}(BA_5) &\cong \mathbb{Z}^2,
\end{aligned}$$

and in both cases, $\Omega_5^{\mathrm{Spin}^c}$ *is torsion. Hence both* $Ph_0^{A_5}(\mathbb{R}^3; f_0^A)$ *and* $Ph_0^{A_5}(\mathbb{R}^3; f_{1/2}^A)$ *vanish.*

Proof. The calculation separates into 2-primary and odd-primary computations; by Lemma 4.5.102, the 2-primary pieces are exactly as in Theorem 4.5.8.

The calculation of the odd-primary parts follows the same line of reasoning as the proof of Lemma 4.5.75: as usual, use the odd-primary equivalence $MTSpin^c \to MTSO \wedge (B\mathrm{U}_1)_+$. We know from Proposition 4.5.103 that the only odd-primary torsion in $H_k(BA_5)$ for $k \le 6$ is $\mathbb{Z}/15 \subset H_3$; feeding that to the Künneth formula, the only odd-primary torsion in $H_k(B\mathrm{U}_1 \times BA_5)$ is two $\mathbb{Z}/15$ summands in H_3 and H_5. Then the Atiyah-Hirzebruch argument is identical to the argument in Lemma 4.5.75. $\square$

4.5.7. Full icosahedral symmetry. If one includes orientation-reversing symmetries of the icosahedron, the symmetry group enlarges to $A_5 \times \mathbb{Z}/2$, with the $\mathbb{Z}/2$ generated by an inversion. This symmetry group is also incompatible with translations, so Remark 4.5.101 applies. This calculation also quickly reduces to something we already know: restricting the representation to $A_4 \times \mathbb{Z}/2$ yields the pyritohedral symmetry representation we studied in §4.5.2.

Theorem 4.5.108. *Let* ρ *be a virtual* $A_5 \times \mathbb{Z}/2$*-representation with rank zero, and let* $V_\rho \to BG$ *denote the associated virtual vector bundle. Suppose that* $w_1(V_\rho) = x$*, where* x *denotes the generator of* $H^1(B\mathbb{Z}/2; \mathbb{Z}/2) \subset H^1(B(A_5 \times \mathbb{Z}/2); \mathbb{Z}/2)$*. Then inclusion of the pyritohedral symmetry subgroup* $\varphi\colon A_4 \times \mathbb{Z}/2 \hookrightarrow A_5 \times \mathbb{Z}/2$ *induces a homotopy equivalence* $B(A_4 \times \mathbb{Z}/2)^{V_\rho} \xrightarrow{\simeq} B(A_5 \times \mathbb{Z}/2)^{V_\rho}$*.*

Proof. By the Whitehead theorem, it suffices to establish that φ induces an isomorphism $\widetilde{H}^*(B(A_5 \times \mathbb{Z}/2)^{V_\rho}; k) \to \widetilde{H}^*(B(A_4 \times \mathbb{Z}/2)^{V_\rho}; k)$ for $k = \mathbb{Q}$ and $k = \mathbb{Z}/p$ for all primes p.

Lemma 4.5.102 and the Künneth theorem imply that $\varphi^*\colon H^*(B(A_5 \times \mathbb{Z}/2); \mathbb{Z}/2) \to H^*(B(A_4 \times \mathbb{Z}/2); \mathbb{Z}/2)$ is an isomorphism. Together with the Thom isomorphism theorem, this takes care of the case $k = \mathbb{Z}/2$.

Let G be either of $A_4 \times \mathbb{Z}/2$ or $A_5 \times \mathbb{Z}/2$; the map $B\varphi\colon B(A_4 \times \mathbb{Z}/2) \to B(A_5 \times \mathbb{Z}/2)$ allows us to think of V_ρ as over BG for either G, and make sense of the statement $w_1(V_\rho) = x$. The Thom isomorphism implies $\widetilde{H}^*((BG)^{V_\rho}; \mathbb{Z}) \cong H^*(BG; \mathbb{Z}_x)$, and since $\mathbb{Z}_x$ arises as a pullback local system along $BG \to B\mathbb{Z}/2$, the twisted Künneth formula proves $\widetilde{H}^*(BG; \mathbb{Z})$ is 2-torsion. The universal coefficient theorem then implies that when we take coefficients in $k = \mathbb{Q}$ or $k = \mathbb{Z}/p$ for p odd, $\widetilde{H}^*(B(A_4 \times \mathbb{Z}/2)^{V_\rho}; k)$ and $H^*(B(A_5 \times \mathbb{Z}/2)^{V_\rho}; k)$ vanish, so the map between them is vacuously an isomorphism. $\square$

Let $\lambda\colon A_5 \times \mathbb{Z}/2 \to \mathrm{O}_3$ denote the representation as the group of symmetries of an icosahedron and $V_\lambda \to B(A_5 \times \mathbb{Z}/2)$ denote the associated vector bundle. Then $w_1(V_\lambda) = x$. Let f_0^D and $f_{1/2}^D$ denote the spinless, resp. spin-1/2 class D equivariant local systems of symmetry types, and f_0^A and $f_{1/2}^A$ denote their analogues in class A.

Corollary 4.5.109. *φ induces homotopy equivalences*

$$B(A_4 \times \mathbb{Z}/2))^{3-V_\lambda} \xrightarrow{\cong} (B(A_5 \times \mathbb{Z}/2))^{3-V_\lambda} \tag{4.5.110a}$$

$$(B(A_4 \times \mathbb{Z}/2))^{\mathrm{Det}(V_\lambda)-1} \xrightarrow{\cong} (B(A_5 \times \mathbb{Z}/2))^{\mathrm{Det}(V_\lambda)-1}. \tag{4.5.110b}$$

Therefore

(1) Proposition 4.5.12 implies that $Ph_0^{A_5\times\mathbb{Z}/2}(\mathbb{R}^3; f_0^D) \cong (\mathbb{Z}/2)^{\oplus 3}$;
(2) Theorem 4.5.26 implies that $Ph_0^{A_5\times\mathbb{Z}/2}(\mathbb{R}^3; f_{1/2}^D) \cong \mathbb{Z}/2$;
(3) Theorem 4.5.28 implies that $Ph_0^{A_5\times\mathbb{Z}/2}(\mathbb{R}^3; f_0^A) \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{\oplus 3}$; and
(4) Theorem 4.5.30 implies that $Ph_0^{A_5\times\mathbb{Z}/2}(\mathbb{R}^3; f_{1/2}^A) \cong \mathbb{Z}/8 \oplus (\mathbb{Z}/2)^{\oplus 3}$.

4.6. Glide symmetry protected phases

Though we have focused on point group symmetries thus far, Freed-Hopkins' ansatz [**FH19a**] also applies to crystallographic groups. In this section, we apply their ansatz to the group of glide symmetries; invertible phases equivariant for this symmetry have been studied by Lu-Shi-Lu [**LSL17**] and Xiong-Alexandradinata [**XA18**], and our results agree with theirs. In particular, Lu-Si-Lu make a conjecture classifying certain glide-symmetric phases in all symmetry types, and we prove that their conjecture follows from Freed-Hopkins' ansatz.

The group of *glide symmetries* acting on $\mathbb{R}^d$, $d \geq 2$, is the free group on the single generator

$$(x_1, x_2, \ldots, x_d) \mapsto (x_1 + 1, -x_2, x_3, \ldots, x_d). \tag{4.6.1}$$

In previous sections, when the symmetry type is $H = \mathrm{Spin}$, Spin^c, $\mathrm{Pin}^\pm$, etc., the symmetry type can mix with the group action on spacetime, corresponding physically to spinless or spin-1/2 fermions. Here, this cannot happen: if μ_2 denotes the kernel of the map $\mathrm{Spin}_n \to \mathrm{SO}_n$ or $\mathrm{Pin}_n^\pm \to \mathrm{O}_n$, all extensions

$$0 \longrightarrow \mu_2 \longrightarrow \widetilde{G} \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{4.6.2}$$

split, so given one of these symmetry types, there is a unique equivariant symmetry type for this $\mathbb{Z}$-action with respect to mixing with fermion parity, corresponding to the trivial local system $\underline{E} \to \mathbb{R}^d$ with value $E \coloneqq \mathrm{Map}(MTH, \Sigma^2 I_{\mathbb{Z}})$.

Definition 4.6.3. Recall from Remark 4.1.25 that we defined a "forgetful map" $\varphi\colon Ph_*^{\mathbb{Z}}(\mathbb{R}^d;\underline{E}) \to Ph_*(\mathbb{R}^d;\underline{E})$. The *intrinsically $\mathbb{Z}$-equivariant phase homology*, denoted $\widehat{Ph}_*^{\mathbb{Z}}(\mathbb{R}^d;\underline{E})$, is the kernel of this map.

This corresponds under Freed-Hopkins' ansatz to what Lu-Shi-Lu call a *glide SPT*: an invertible phase equivariant for a $\mathbb{Z}$ glide symmetry which is trivializable when one forgets the symmetry.

Let $\mathrm{TP}_d(H)$ denote the abelian group of SPT phases in (spatial) dimension d; then Freed-Hopkins' ansatz [**FH16a**] classifying these phases in terms of invertible field theories predicts $\mathrm{TP}_d(H) \cong E_{-d}$.

Lu-Shi-Lu [**LSL17**] studied groups of glide SPTs and conjectured a formula classifying them in terms of the classification of ordinary SPTs. We prove the corresponding statement on phase homology groups.

Theorem 4.6.4. *For a given symmetry type $\rho_n\colon H_n \to \mathrm{O}_n$, there is a natural isomorphism $\widehat{Ph}_0^{\mathbb{Z}}(\mathbb{R}^d;\underline{E}) \cong E_{-(d-1)} \otimes \mathbb{Z}/2$.*

Passing this through the ansatz, this predicts that the group of glide SPTs is naturally isomorphic to $\mathrm{TP}_{d-1}(H) \otimes \mathbb{Z}/2$, which is Lu-Shi-Lu's original conjecture [**LSL17**, Conjecture 1]. In addition, Xiong-Alexandradinata [**XA18**] obtain this result using physics-based arguments.

Proof of Theorem 4.6.4. We calculate the 0^{th} $\mathbb{Z}$-equivariant Borel-Moore E-homology of $\mathbb{R}^d$. As the $\mathbb{Z}$-action is free, this is the 0^{th} (nonequivariant) Borel-Moore E-homology of the fundamental domain $X \coloneqq \mathbb{R}^d/\mathbb{Z}$. Since the one-point compactification $\overline{X}$ of X is a finite CW complex, this Borel-Moore homology is isomorphic to $\widetilde{E}_0(\overline{X})$.

If $\sigma \to S^1$ denotes the Möbius bundle, then X is diffeomorphic to the total space of $\sigma \oplus \underline{\mathbb{R}}^{d-2} \to S^1$, so $\overline{X}$ is the Thom space $(S^1)^{\sigma+d-2}$. The identification $(S^1)^{\sigma} \cong \mathbb{RP}^2$ induces $\overline{X} \cong \Sigma^{d-2}\mathbb{RP}^2$, and therefore

$$Ph_*^{\mathbb{Z}}(\mathbb{R}^d;\underline{E}) \cong \widetilde{E}_0(\Sigma^{d-2}\mathbb{RP}^2) \cong \widetilde{E}_{2-d}(\mathbb{RP}^2). \tag{4.6.5}$$

Lemma 4.6.6. *Let $p\colon S^2 \to \mathbb{RP}^2$ be the double cover map and $s\colon \widetilde{E}_k(S^1) \to \widetilde{E}_{k+1}(S^2)$ be the suspension isomorphism. The composition $p_* \circ \delta \circ s\colon \widetilde{E}_{-1}(S^2) \to \widetilde{E}_{-1}(S^2)$ is multiplication by 2.*

Proof. This follows because the suspension is the cofiber of the cofiber; then one explicitly checks what happens on mapping cylinders. ☐

Lemma 4.6.7. *Under these isomorphisms, the forgetful map $Ph_0^{\mathbb{Z}}(\mathbb{R}^d;\underline{E}) \to Ph_0(\mathbb{R}^d;\underline{E})$ is identified with δ.*

Proof. Because $\mathbb{Z}$ acts freely on $\mathbb{R}^d$, $E_{0,\mathrm{BM}}^{\mathbb{Z}}(\mathbb{R}^d)$ is identified with $\widetilde{E}_0$ of the one-point compactification of $\mathbb{R}^d/\mathbb{Z}$, which we saw above is homeomorphic to $\Sigma^{d-2}\mathbb{RP}^2$. The codomain of the forgetful map is $E_{0,\mathrm{BM}}(\mathbb{R}^d) \cong$

$\widetilde{E}_0(\Sigma^{d-2}S^2)$, so we have identified δ with a map $\widetilde{E}_0(\Sigma^{d-2}\mathbb{RP}^2) \to \widetilde{E}_0(\Sigma^{d-2}S^2)$. But tracing through the construction in Remark 4.1.25, this map comes from applying $\widetilde{E}_0$ to an actual map $\Sigma^{d-2}\mathbb{RP}^2 \to \Sigma^{d-2}S^2$.

Next, precompose with $\Sigma^{d+2}p\colon \Sigma^{d-2}S^2 \to \Sigma^{d-2}\mathbb{RP}^2$ and check that this map has degree 2, agreeing with Lemma 4.6.6. This suffices to identify the maps because $p^*\colon [\mathbb{RP}^2, S^2] \to [S^2, S^2]$ is injective. $\square$

$\mathbb{RP}^2$ is homeomorphic to the cofiber of a degree-2 map $S^1 \to S^1$. Hence there is a long exact sequence in reduced E-homology

$$\cdots \longrightarrow \widetilde{E}_{2-d}(S^1) \xrightarrow{m} \widetilde{E}_{2-d}(S^1) \xrightarrow{r} \widetilde{E}_{2-d}(\mathbb{RP}^2) \xrightarrow{\delta} \widetilde{E}_{1-d}(S^1) \longrightarrow \cdots \tag{4.6.8}$$

where m is multiplication by 2. Exactness implies $\ker(\delta) = \mathrm{Im}(r) = \mathrm{coker}(m)$. Using the suspension isomorphism, $\widetilde{E}_k(S^1) \cong \widetilde{E}_{k-1}$, and therefore $\mathrm{coker}(m) \cong E_{-(d-1)} \otimes \mathbb{Z}/2$, and 4.6.7 identifies δ with the forgetful map from equivariant to nonequivariant phase homology for $\mathbb{R}^d$. In particular, $\widehat{Ph}{}_0^{\mathbb{Z}}(\mathbb{R}^d; \underline{E}) \cong \ker(\delta)$, which we have naturally identified with $E_{-(d-1)} \otimes \mathbb{Z}/2$. $\square$

Remark 4.6.9. Using the long exact sequence (4.6.8), we observe that $Ph_0^{\mathbb{Z}}(\mathbb{R}^d; \underline{E})$ has exponent 4. This is because for any long exact sequence of abelian groups

$$\cdots \longrightarrow A \xrightarrow{\cdot 2} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\cdot 2} C \longrightarrow \cdots \tag{4.6.10}$$

in which A and C are finitely generated, $\mathrm{Im}(f) \cong A/2$, hence has exponent 2, and $\ker(g)$ is isomorphic to the subgroup of order-2 elements of C, which also has exponent 2. Since B is an extension of $\ker(g)$ by $\mathrm{Im}(f)$, B has exponent 4.

Passing this observation through Freed-Hopkins' ansatz, this recovers an observation of Xiong-Alexandradinata [**XA18**]: that *any* phase equivariant with respect to glide symmetry, whether a glide SPT or not, has order dividing 4.

Example 4.6.11. In Altland-Zirnbauer class AII, corresponding to the symmetry type $\mathrm{pin}^{\tilde{c}+}$, the ansatz predicts a unique nontrivial glide SPT in dimension $2+1$, coming from the classification

$$[MTPin^{\tilde{c}+}, \Sigma^4 I_{\mathbb{Z}}] \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2 \tag{4.6.12}$$

(the calculation of $[MTPin^{\tilde{c}+}, \Sigma^4 I_{\mathbb{Z}}]$ is due to Freed-Hopkins [**FH16a**, §9.3]). Physicists are particularly interested in this nontrivial glide SPT phase, which is predicted to have unusual surface states called "hourglass fermions" [**WACB16**], and which has been studied experimentally [**MYL$^+$17**].

4.7. Conclusion and outlook

We conclude by indicating a few directions of potential further research.

4.7.1. From free fermions to interacting phases. Free fermion phases are a rich source of examples of invertible phases in the physics literature, at least for symmetry types spin, pin$^\pm$, spin^c, etc. The classification of free fermion systems uses K-theory: see Kitaev [**Kit09**] for the original proposal and Freed-Moore [**FM13**] for a comprehensive classification. However, for a given dimension and symmetry type, the map from free fermion systems to invertible phases of matter can in general have both kernel (as first observed by Fidkowski-Kitaev [**FK10, FK11**] and Turner-Pollmann-Berg [**TPB11**]) and cokernel (as first observed by Wang-Potter-Senthil [**WPS14**] and Wang-Senthil [**WS14**]). Researchers are also interested in the free-to-interacting map for phases with spatial symmetries, and this map has been studied from a physics point of view for crystalline phases in several works, including [**YR13, IF15, MFM15, LTH16, SS17, RL18, Zou18, LVK19, RS20, ZYQG20, ACR$^+$21**].

Freed-Hopkins [**FH16a**, §9.2, §9.3] mathematically model the map from free to interacting systems using the Atiyah-Bott-Shapiro map $\mathit{MTSpin} \to KO$ [**ABS64**], but they do not consider spatial symmetries. In view of the large bodies of research on free fermions with spatial symmetries and invertible phases with spatial symmetries, it would be nice to understand the map between them in the presence of spatial symmetry from the low-energy field theory perspective, and to make contact with the work of Adem, Antolín Camarena, Semenoff, and Sheinbaum [**AACSS16**], Sheinbaum and Antolín Camarena [**SC20**], and Cornfeld-Carmeli [**CC21**] studying free fermion phases with spatial symmetries using methods from homotopy theory. This is something we hope to tackle in future work.

4.7.2. Other symmetry types. We investigated two of the ten Altland-Zirnbauer classes, and it would be interesting to know whether a version of the FCEP holds for other classes. One starting point could be class C, corresponding to a spin^h structure [**FH16a**, (9.25)];[33] the calculations in §4.2.8 could be applied to Spin_n^h to obtain a fermionic crystalline equivalence principle for class C and hopefully phase homology calculations predicting the existence of additional crystalline SPT phases.

Several teams of researchers have studied or classified interacting fermionic crystalline SPTs for other Altland-Zirnbauer types, including [**YR13, YX14, CHMR15, LTH16, WF17, CW18, RL18, SXG18, MSH19, ZXXS20, ZYQG20**]. It would be good to compare their computations with the predictions made by an FCEP in other symmetry types.

[33] Spin^h is the symmetry type $\mathrm{Spin} \times_{\mu_2} \mathrm{SU}_2 \to \mathrm{O}$. Freed-Hopkins [**FH16a**, Proposition 9.16] call this symmetry type G^0; it is sometimes also called spin-SU_2, e.g. in [**WWW19**]. Likewise, the symmetry types pin$^{h\pm}$ we refer to later in this section are defined to be $\mathrm{Pin}^\pm \times_{\mu_2} \mathrm{SU}_2$, and are called $G^\pm$ by Freed-Hopkins [**FH16a**, Proposition 9.16].

Another interesting potential connection with preexisting work is the case of class A phases with a spatial reflection interacting with the internal U_1 symmetry. Depending on how the symmetries mix, Shiozaki-Shapourian-Gomi-Ryu [**SSGR18**, §V.C, §V.E] and Thorngren-Else [**TE18**, §VII.B] obtain classifications in terms of $\mathrm{pin}^{\tilde{c}\pm}$ bordism, and we would be interested in knowing whether that can also be obtained from our ansatz. Similarly, can one begin with class C phases and a reflection acting on the internal SU_2 symmetry and obtain a classification in terms of $\mathrm{pin}^{h\pm}$ bordism?

4.7.3. Crystallographic groups. Though we discussed glide symmetries in §4.6, we have barely touched upon the rich world of crystallographic groups. Free-fermion phases equivariant for these groups have been studied, e.g. in [**SMJZ13, KdBvW$^+$17, SSG18, OSS19**], but much less is known about the interacting case, even though the our ansatz applies to it. There are some classifications by other methods for various classes of crystallographic groups; for example, Ouyang-Wang-Gu-Qi [**OWGQ20**] study wallpaper group symmetries, and Sheinbaum-Antolín Camarena [**SC20**] provide a general framework and a few examples. There is also work by Wang-Alexandradinata-Cava-Bernevig [**WACB16**] and Guo-Ohmori-Putrov-Wan-Wang [**GOP$^+$20**] studying interacting phases for specific crystallographic groups that are not point groups.

4.7.4. Lattice realizations. Modeling topological phases as lattice Hamiltonian systems can make any crystallographic symmetries acting on space very explicit, using a lattice and Hamiltonian invariant under the symmetry of interest. Our predictions of point group SPTs should correspond to actual lattice models of phases. We listed several specific predicted phases of interest in §4.3.1, and these would make for good starting points for lattice realizations.

CHAPTER 5

Stable diffeomorphism classification of some unorientable 4-manifolds

The content of this chapter appears on the ArXiv as the preprint [**Deb21b**]. It has been slightly edited to be streamlined with the rest of the thesis.

5.0. Introduction

The classification of closed 4-manifolds up to diffeomorphism is impossible in general: a solution would also solve the word problem for groups. Even if one fixes the fundamental group to avoid this problem, the classification is still currently intractable. For this reason, topologists study weaker classifications of 4-manifolds which are coarse enough to be calcuable yet fine enough to be useful.

Stable diffeomorphism is an example of such an invariant. Two closed 4-manifolds M and N are *stably diffeomorphic* if there are $m, n \geq 0$ such that $M \# m(S^2 \times S^2)$ is diffeomorphic to $N \# n(S^2 \times S^2)$. This notion of equivalence has applications to quantum topology: for example, Reutter [**Reu20**, Theorem A] shows that the partition functions of 4d semisimple oriented TFTs are insensitive to stable diffeomorphism along the way to showing that such TFTs cannot distinguish homotopy-equivalent closed, oriented 4-manifolds. And stable diffeomorphism classes are computable: once the fundamental group G is fixed, Kreck [**Kre99**] shows how to reduce the classification of 4-manifolds up to stable diffeomorphism to a collection of bordism computations, and for many choices of G, the classification of closed, connected, oriented 4-manifolds with $\pi_1(M) \cong G$ up to stable diffeomorphism has been completely worked out, thanks to work of Wall [**Wal64**], Teichner [**Tei92**], Spaggiari [**Spa03**], Crowley-Sixt [**CS11**], Politarczyk [**Pol13**], Kasprowski-Land-Powell-Teichner [**KLPT17**], Pedrotti [**Ped17**], Hambleton-Hildum [**HH19**], and Kasprowski-Powell-Teichner [**KPMT20**].

Researchers interested in topological manifolds also study *stable homeomorphism* of topological manifolds, i.e. homeomorphism after connect-summing with some number of copies of $S^2 \times S^2$. Kreck's theorem applies to this case too, reframing the question in terms of bordism of topological manifolds. Stable homeomorphism classifications are studied by Teichner [**Tei92**, §5], Wang [**Wan95**], Hambleton-Kreck-Teichner [**HKT09**], Kasprowski-Land-Powell-Teichner [**KLPT17**, §§4–5], Hambleton-Hildum [**HH19**], and Kasprowski-Powell-Teichner [**KPMT20**, §2.3], Much less work has been done on unorientable 4-manifolds, even though the theory still works and is simpler in some cases, as we explain below. There is some work in the literature, such as that of Kreck [**Kre84**], Wang [**Wan95**], Kurazono [**Kur01**], Davis [**Dav05**], and Friedl-Nagel-Orson-Powell [**FNOP19**, §12].

The goal of this paper is to compute sets of stable diffeomorphism and stable homeomorphism classes for a class of unorientable 4-manifolds, as well as determining the corresponding complete stable diffeomorphism and homeomorphism invariants. As a consequence of our Theorem 5.2.1, for many finite groups G, the classification of stable diffeomorphism or homeomorphism classes of unorientable 4-manifolds with $\pi_1(M) \cong G$ reduces to the stable classifications for a smaller 2-group. For example, we show that the stable diffeomorphism, resp. homeomorphism classification when $\pi_1(M) \cong \mathbb{Z}/2$ determines the stable diffeomorphism, resp. homeomorphism classification for any G of order 2 mod 4. We then compute these classifications using Kreck's techniques.

Suppose G is the fundamental group of an unorientable manifold. Then there is an extension

$$1 \longrightarrow K \longrightarrow G \longrightarrow \mathbb{Z}/2 \longrightarrow 1, \tag{5.0.1}$$

where $G \twoheadrightarrow \mathbb{Z}/2$ is defined by classifying loops as orientation-preserving or orientation-reversing. Therefore $\mathbb{Z}/2$ acts on K.

Theorem (Main theorem). *Let G be a finite group of order* 2 mod 4*, and suppose that in* (5.0.1)*,* $\mathbb{Z}/2$ *acts trivially on* $H^*(BK)$*.*

(1) There are fourteen equivalence classes of closed, connected, unorientable 4*-manifolds M up to stable diffeomorphism: nine for which M is pin$^+$, one for which M is pin$^-$, and four for which M is neither.*

(2) There are twenty equivalence classes of closed, connected, unorientable topological 4*-manifolds M up to stable homeomorphism: ten for which M is pin$^+$, two for which M is pin$^-$, and eight for which M is neither.*

This is a combination of Theorems 5.3.2, 5.3.5, 5.4.2 and 5.4.5. In those theorems we also determine complete stable diffeomorphism/homeomorphism invariants for these manifolds. The classification for M neither pin$^+$ or pin$^-$ can be extracted from work of Davis [**Dav05**, Theorem 2.3], but the other parts are new.

We prove these theorems by establishing isomorphisms of bordism groups. Specifically, Kreck's modified surgery theory associates to G a set of symmetry types $\xi\colon B \to B\mathrm{O}$ and expresses the set of stable diffeomorphism classes in terms of the bordism groups Ω_4^ξ; we show that when $|G| \equiv 2 \bmod 4$, the Thom spectra of these symmetry types are homotopy equivalent to the Thom spectra for unoriented, pin$^+$, and pin$^-$

bordism. In the smooth case, the bordism groups Ω_4^{O}, $\Omega_4^{\mathrm{Pin}^+}$, and $\Omega_4^{\mathrm{Pin}^-}$ are well-known. The topological versions of these bordism groups are less well-known, but Kirby-Taylor [**KT90b**, §9] compute $\Omega_4^{\mathrm{TopPin}^\pm}$ and provide enough information for us to compute Ω_4^{Top}, which we do in Proposition 5.4.7.

The argument we use to establish the isomorphism from ξ-bordism to a simpler kind of bordism applies to more general choices of $\pi_1(M)$.

Theorem 5.2.1. *Suppose G is a finite group fitting into an extension*

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} P \longrightarrow 1, \tag{5.0.2}$$

where $|K|$ is odd and P is a 2-group, and suppose P acts trivially on $H^*(BK)$. For any unorientable virtual vector bundle $V \to BP$, φ induces an equivalence of Thom spectra $(BG)^{\varphi^* V} \xrightarrow{\simeq} (BP)^V$.

The Pontrjagin-Thom construction turns this equivalence into isomorphisms of bordism groups from the unorientable symmetry types Kreck associates to G to the unorientable symmetry types for P, which we can use to compute stable diffeomorphism classes. The proof strongly requires the assumption that V is unorientable; nothing like this is true in the oriented case.

Our main theorem above covers the case $|G| \equiv 2 \bmod 4$. The next step would be to consider $P \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ or $\mathbb{Z}/4$, which would suffice for many groups G of order $4 \bmod 8$. For these choices of P, many of the needed bordism groups have already been computed in the literature for other applications. For $P \cong \mathbb{Z}/4$, see Botvinnik-Gilkey [**BG97**, §5]; for $P \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, see §4.4.4 and work of Guo-Ohmori-Putrov-Wan-Wang [**GOP$^+$20**, §7], the author in [**KPMT20**, Appendix F], and Wan-Wang-Zheng [**WWZ20**, Appendix A].

We begin in §5.1 with a quick review of Kreck's theorem [**Kre99**] on stable diffeomorphism classes of 4-manifolds within a given 1-type. In §5.2, we study the Thom spectra of unorientable vector bundles over BG, where G is a finite group, proving Theorem 5.2.1. In §5.3, we specialize to the case where $|G| \equiv 2 \bmod 4$, determining the three possible normal 1-types and computing the sets of stable diffeomorphism classes for them. We prove Theorems 5.3.2 and 5.3.5, which together form the smooth part of the main theorem above. In Example 5.3.4, we discuss an example: $\mathbb{RP}^4$ is homeomorphic but not stably diffeomorphic to Cappell-Shaneson's fake $\mathbb{RP}^4$. This fact was known to Cappell-Shaneson [**CS71, CS76**] and the proof using Kreck's surgery theory is due to Stolz [**Sto88**]. In §5.4, we consider stable homeomorphism classes of topological manifolds with $|\pi_1(M)| \equiv 2 \bmod 4$, and prove Theorems 5.4.2 and 5.4.5, which form the topological part of the main theorem above.

5.1. Review: normal 1-types, normal 1-smoothings, and stable diffeomorphism classes

We review some standard definitions in this area. We will always assume our manifolds are closed and connected. Except in §5.4, we also assume they are smooth.

Definition 5.1.1. A *normal* 1-*type* of a manifold M is a symmetry type $\xi\colon B \to B\mathrm{O}$ such that there is a lift of the map $\nu\colon M \to BO$ classifying the stable normal bundle of M to a map $\widetilde{\nu}\colon M \to B$ such that $\xi \circ \widetilde{\nu} = \nu$, $\widetilde{\nu}$ is 2-connected, and ξ is 2-coconnected.

A choice of such a lift is called a *normal* 1-*smoothing* of M.

Any two normal 1-types of a given manifold are homotopy equivalent as spaces over $B\mathrm{O}$, so we will abuse notation and say "the" normal 1-type.

The map $\xi\colon B \to B\mathrm{O}$ determines a bordism theory Ω_*^ξ of manifolds with a lift of the stable normal bundle across ξ; a normal 1-smoothing of M determines a class in this bordism group. Different normal smoothings of the same manifold do not always define the same class in Ω_*^ξ.

Let $V_{\mathrm{SO}} \to B\mathrm{SO}$, $V_{\mathrm{Spin}} \to B\mathrm{Spin}$, etc., denote the tautological stable vector bundles over their respective spaces. We use the convention that maps to $B\mathrm{O}$ are represented by rank-zero virtual vector bundles, which is why we write $E - \dim E$ in (5.1.3), for example.

Example 5.1.2 (Kreck [Kre99, §2, Proposition 2])**.** When M is unorientable, Kreck classifies the possible normal 1-types of M into two families. Let $M' \to M$ be the universal cover of M, which is classified by a map $\theta\colon M \to B\pi_1(M)$.

Almost spin: If M' admits a spin structure, M is called *almost spin*. In this case, $w_1(M) = \theta^* x_1$ and $w_2(M) = \theta^* x_2$ for some $x_1, x_2 \in H^*(BG;\mathbb{Z}/2)$. Assume there is a vector bundle $E \to BG$ such that $w_i(E) = x_i$ for $i = 1, 2$.[1] Then, the normal 1-type of M is

(5.1.3)
$$\begin{tikzcd} & B\mathrm{Spin} \times B\pi_1(M) \arrow[d, "V_{\mathrm{Spin}} \oplus (E - \dim E)"] \\ M \arrow[ur] \arrow[r, "\nu"'] & B\mathrm{O}. \end{tikzcd}$$

Totally non-spin: If M' does not admit a spin structure, M is called *totally non-spin*. In this case, $w_1(M) = \theta^* x$ for some $x \in H^1(BG;\mathbb{Z}/2)$. Let $E \to BG$ be a line bundle with $w_1(E) = x$. Then

[1] This will be true for all cases we consider in this paper, but is not true in general.

the normal 1-type of M is

$$\begin{tikzcd} & B\mathrm{SO} \times B\pi_1(M) \arrow[d, "V_{\mathrm{SO}} \oplus (E-1)"] \\ M \arrow[ur] \arrow[r, "\nu"'] & B\mathrm{O}. \end{tikzcd} \tag{5.1.4}$$

Because $S^2 \times S^2$ has trivial stable normal bundle, taking connect sum with $S^2 \times S^2$ does not change the normal 1-type of a 4-manifold; thus the classification of 4-manifolds up to stable diffeomorphism can proceed one normal 1-type at a time. Moreover, because $S^2 \times S^2$ is null-bordant, one might conclude that stably diffeomorphic 4-manifolds M and N are bordant — or, more precisely, that M and N admit normal 1-smoothings which are bordant in Ω_4^ξ. So a plausible lower bound for the set of stable diffeomorphism classes with normal 1-type ξ would be Ω_4^ξ modulo some identifications arising from inequivalent normal 1-smoothings of the same underlying manifold. Remarkably, this turns out to be a complete classification!

Theorem 5.1.5 (Kreck [**Kre99**, Theorem C; §3, Proposition 4])**.**

(1) If M and N are 4-manifolds of the same normal 1-type ξ admitting normal 1-smoothings which are bordant in Ω_4^ξ, then M is stably diffeomorphic to N.

(2) If $\pi_1(\xi)$ is finite, every class in Ω_4^ξ can be realized as the normal 1-smoothing of a 4-manifold with normal 1-type ξ.

The upshot is that if $\mathrm{Aut}(\xi)$ denotes the group of fiber homotopy equivalences of $\xi \to B\mathrm{O}$, the set of stable diffeomorphism classes of 4-manifolds with normal 1-type ξ is $\Omega_4^\xi/\mathrm{Aut}(\xi)$.

The set of bordism classes of normal 1-smoothings of a given 4-manifold is contained within an $\mathrm{Aut}(\xi)$-orbit of Ω_4^ξ, so one effect of the quotient is to identify these as all coming from the same manifold.

This illustrates the standard way to calculate stable diffeomorphism classes: determine Ω_4^ξ, then determine the $\mathrm{Aut}(\xi)$-action. These bordism groups are the homotopy groups of the Thom spectrum $M\xi$ of ξ, so in the next section we begin the calculation of stable diffeomorphism classes by simplifying $M\xi$.

5.2. Simplifying Thom spectra

Theorem 5.1.5 tells us to investigate the Thom spectra of the normal 1-types in Example 5.1.2. In both cases, the vector bundle is an exterior direct sum, so the Thom spectra split, as $\mathit{MSpin} \wedge (B\pi_1(M))^V$ in the almost spin case and $\mathit{MSO} \wedge (BG)^V$ in the totally non-spin case, where V is a rank-zero unoriented virtual vector bundle. We attack the problem by simplifying $(B\pi_1(M))^V$ for some choices of $\pi_1(M)$.

Theorem 5.2.1. *Suppose G is a finite group fitting into an extension*

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\varphi} P \longrightarrow 1, \tag{5.2.2}$$

where $|K|$ is odd and P is a 2-group, and suppose P acts trivially on $H^*(BK)$. For any unorientable virtual vector bundle $V \to BP$, φ induces an equivalence of Thom spectra $(BG)^{\varphi^*V} \xrightarrow{\simeq} (BP)^V$.

We'll prove this in a series of lemmas. Recall from Definition 1.1.24 that if H is a group, A is an abelian group, and $\alpha \in H^1(BH; \mathbb{Z}/2)$, A_α denotes the $\mathbb{Z}[H]$-module which is the abelian group $\mathbb{Z}$ with the H-action in which $g \in H$ acts by $(-1)^{\alpha(g)}$. Here we use the identification $H^1(BH; \mathbb{Z}/2) \cong \mathrm{Hom}(H, \mathbb{Z}/2)$.

Lemma 5.2.3. *In the situation of Theorem 5.2.1, both $\widetilde{H}^*((BG)^{\varphi^*V})$ and $\widetilde{H}^*((BP)^V)$ are 2-torsion.*

Proof. Define the $\mathbb{Z}[P]$-module $A_{w_1(V)}$ and the $\mathbb{Z}[G]$-module $A_{w_1(\varphi^*P)}$, which is isomorphic to the pullback of $A_{w_1(V)}$ by φ. The Thom isomorphism provides isomorphisms of graded abelian groups

$$H^*(BP; \mathbb{Z}_{w_1(V)}) \xrightarrow{\cong} \widetilde{H}^*((BP)^V) \tag{5.2.4a}$$

$$H^*(BG; \mathbb{Z}_{w_1(\varphi^*V)}) \xrightarrow{\cong} \widetilde{H}^*((BG)^{\varphi^*V}), \tag{5.2.4b}$$

so we will prove the lemma using group cohomology – specifically, using the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(BP; (H^q(BK; \mathbb{Z}))_{w_1(V)}) \Longrightarrow H^{p+q}(BG; \mathbb{Z}_{w_1(\varphi^*V)}). \tag{5.2.5}$$

Here it is crucial that P acts trivially on $H^*(BK)$; otherwise we would have a different local coefficient system than $(H^q(BK; \mathbb{Z})_{w_1(V)}$ in (5.2.5).

Since $E_2^{p,q} \cong H^p(BP; M_q)$ for some $\mathbb{Z}[P]$-module M_q, $E_2^{p,q}$ is 2-torsion for $p > 1$ by Maschke's theorem.[2] When $p = 0$,

$$E_2^{0,q} \cong H^0(BP; H^q(BK)_{w_1(V)}) \cong (H^q(BK)_{w_1(V)})^P. \tag{5.2.6}$$

We will show this vanishes. First, $H^q(BK)$ is $\mathbb{Z}$ for $q = 0$ and is odd-primary torsion for $q > 0$ (by Maschke's theorem, because $2 \nmid \#K$). Therefore if $a \in H^q(BK)$ and $-a = a$, $a = 0$. Since $w_1(V) \neq 0$, there is some

[2]We use Maschke's theorem as follows: if G is a finite group and k is a field of characteristic 0 or characteristic $\ell \nmid \#G$, the category of $k[G]$-modules is semisimple. Therefore all positive-degree Ext groups vanish, in particular $H^m(BG; M) \cong \mathrm{Ext}^m_{k[G]}(\mathbb{Z}, M)$ for any $k[G]$-module M and $m > 1$. Combined with the universal coefficient theorem, this implies that for any $\mathbb{Z}[G]$-module M and $m > 1$, $H^m(G; M)$ is torsion ($k = \mathbb{Q}$), and lacks ℓ-torsion if $\ell \nmid \#G$.

$g \in P$ which acts on $\mathbb{Z}_{w_1(V)}$ as -1, hence also acts on $H^q(BK)_{w_1(V)}$ as -1, so the subgroup of invariants of $H^q(BK)_{w_1(V)}$ is $\{0\}$.

Considering the line $q = 0$ proves $H^*(BP; \mathbb{Z}_{w_1(V)})$ is 2-torsion. For $H^*(BG; \mathbb{Z}_{w_1(\varphi^*V)})$, we have shown the E_2-page is 2-torsion, so the graded abelian group the spectral sequence converges to is also 2-torsion. $\square$

Lemma 5.2.7. *With G and P as in Theorem 5.2.1, $\varphi^*\colon H^*(BP; \mathbb{Z}/2) \to H^*(BG; \mathbb{Z}/2)$ is an isomorphism of graded rings.*

Proof. Since K has odd order, its mod 2 cohomology is $\mathbb{Z}/2$ in degree 0 and vanishes elsewhere, so the result follows from the Leray-Hirsch theorem applied to the fibration $BK \to BG \to BP$ induced by (5.2.2). $\square$

Proof of Theorem 5.2.1. Use the homology Whitehead theorem: if $f\colon X \to Y$ is a map of bounded-below spectra which induces an isomorphism on rational cohomology and on mod p cohomology for every prime p, then f is a homotopy equivalence. Lemma 5.2.3 and the universal coefficient theorem imply that if $k = \mathbb{Q}$ or $k = \mathbb{Z}/p$ for an odd prime p, $\widetilde{H}^*((BG)^{\varphi^*V}; k)$ and $\widetilde{H}^*((BP)^V; k)$ both vanish, so the map between them is vacuously an isomorphism. The sole remaining case is $p = 2$. Since $1 \equiv -1 \bmod 2$, $(\mathbb{Z}/2)_{w_1(V)}$ carries the trivial P-action; thus, the Thom isomorphism has the form

$$H^*(BP; \mathbb{Z}/2) \xrightarrow{\cong} \widetilde{H}^*((BP)^V; \mathbb{Z}/2). \tag{5.2.8a}$$

Analogously, there is a Thom isomorphism

$$H^*(BG; \mathbb{Z}/2)) \xrightarrow{\cong} \widetilde{H}^*((BG)^{\varphi^*V}; \mathbb{Z}/2). \tag{5.2.8b}$$

As the Thom isomorphism is functorial with respect to pullbacks of vector bundles, Lemma 5.2.7 lifts to imply that

$$\varphi^*\colon \widetilde{H}^*((BP)^V; \mathbb{Z}/2) \longrightarrow \widetilde{H}^*((BG)^{\varphi^*V}; \mathbb{Z}/2) \tag{5.2.9}$$

is an isomorphism. $\square$

5.3. The case $|\pi_1(X)| \equiv 2 \bmod 4$

If M is an unorientable manifold, the description of loops as orientation-preserving or orientation-reversing defines a surjection $p\colon \pi_1(M) \to \mathbb{Z}/2$, so $\pi_1(M)$ cannot have odd order. Thus the simplest case occurs when

$|\pi_1(M)| \equiv 2 \bmod 4$, so that $|\ker(p)|$ is odd. For the rest of this section, fix such a group G, and assume that $\mathbb{Z}/2$ acts trivially on $H^*(B\ker(p))$.

In this case, Theorem 5.2.1 applies to show that if $\mathbb{Z}/2$ acts trivially on $H^*(B\ker(p))$ and $V \to B\mathbb{Z}/2$ is any unorientable virtual vector bundle, the map $(B\pi_1(M))^{p^*V} \xrightarrow{\simeq} (B\mathbb{Z}/2)^V$ is an equivalence.

Let $\sigma \to B\mathbb{Z}/2$ denote the tautological line bundle and $x \coloneqq w_1(\sigma) \in H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$, so $H^*(B\mathbb{Z}/2;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]$. Because $\ker(p)$ has odd order, the Leray-Hirsch theorem implies

$$p^*\colon H^*(B\mathbb{Z}/2;\mathbb{Z}/2) \longrightarrow H^*(B\pi_1(M);\mathbb{Z}/2) \tag{5.3.1}$$

is an isomorphism.

5.3.1. The almost spin case. Example 5.1.2 shows there are two unorientable normal 1-types in this case: $w_1(\nu) \neq 0$, so it must be the pullback of $p^*x \in H^1(B\pi_1(M);\mathbb{Z}/2)$, and for w_2, we have two choices: $w_2 = 0$ (the normal bundle is pin$^+$) and $w_2 = p^*x^2$ (the normal bundle is pin$^-$).

Recall that for a manifold M, M is pin$^\pm$ (i.e. the tangent bundle is pin$^\pm$) iff the normal bundle is pin$^\mp$. A (tangential) pin$^+$ 4-manifold M has a $\mathbb{Z}/16$-valued invariant given by the η-invariant of a twisted Dirac operator [**Sto88**, §4]; let η' be the invariant assigning to a pin$^+$ 4-manifold M the image of this η-invariant in the nine-element set $(\mathbb{Z}/16)/(x \sim -x)$. We will see in the proof of Theorem 5.3.2 that all pin$^+$ structures on M give the same value of η', so we may define it as an invariant of manifolds which admit a pin$^+$ structure, without choosing such a structure.

Theorem 5.3.2. *There are nine stable diffeomorphism classes of unorientable 4-manifolds with $\pi_1(M) \cong G$ that admit a (tangential) pin$^+$ structure, and there is a single stable diffeomorphism class of manifolds with $\pi_1(M) \cong G$ that admit a (tangential) pin$^-$ structure. In the pin$^+$ case, η' is a complete stable diffeomorphism invariant.*

Proof. Both choices of (w_1, w_2) arise from vector bundles: $(p^*x, 0)$ from $p^*\sigma$, and (p^*x, p^*x^2) from $p^*(3\sigma)$. Thus the normal 1-types are

$$V_{\mathrm{Spin}} \oplus (p^*\sigma - 1)\colon B\mathrm{Spin} \times B\pi_1(M) \longrightarrow B\mathrm{O} \tag{5.3.3a}$$

$$V_{\mathrm{Spin}} \oplus (p^*(3\sigma) - 3)\colon B\mathrm{Spin} \times B\pi_1(M) \longrightarrow B\mathrm{O}, \tag{5.3.3b}$$

and their Thom spectra are $\mathit{MSpin} \wedge (B\pi_1(M))^{p^*\sigma - 1}$, resp. $\mathit{MSpin} \wedge (B\pi_1(M))^{p^*(3\sigma)-3}$. By Theorem 5.2.1, these are equivalent to $\mathit{MSpin} \wedge (B\mathbb{Z}/2)^{\sigma-1}$, resp. $\mathit{MSpin} \wedge (B\mathbb{Z}/2)^{3\sigma-3}$. Recall from (4.2.10a) and (4.2.10b) that $MSpin \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq MTPin^-$ and $MSpin \wedge (B\mathbb{Z}/2)^{3\sigma-3} \simeq MTPin^+$. The corresponding bordism groups are known.

- In the case $w_2(\nu) = 0$, $\Omega_4^\xi \cong \Omega_4^{\mathrm{Pin}^-} \cong 0$ [**ABP69, KT90b**] — all 4-manifolds with this normal 1-type are stably diffeomorphic.
- When $w_2(\nu) = p^*x^2$, $\Omega_4^\xi \cong \Omega_4^{\mathrm{Pin}^+} \cong \mathbb{Z}/16$ [**Gia73b, KT90a, KT90b**].

In the latter case we have to determine the $\mathrm{Aut}(\xi)$-action. $\mathbb{RP}^4$ admits two pin$^+$ structures, and Kirby-Taylor [**KT90b**, Theorem 5.2] choose an isomorphism $\Omega_4^{\mathrm{Pin}^+} \xrightarrow{\cong} \mathbb{Z}/16$ sending these two pin$^+$ structures to ± 1. Therefore for any equivalence class $x \in \Omega_4^{\mathrm{Pin}^+}$ and any $g \in \mathrm{Aut}(\mathrm{Pin}^+)$, $g \cdot x = \pm x$, because we can represent x as a disjoint union of copies of $\mathbb{RP}^4$ with some pin$^+$ structure, and the $\mathrm{Aut}(\mathrm{Pin}^+)$-orbit of the $\mathbb{RP}^4$s is $\{\pm 1\}$. The isomorphism from ξ-bordism to pin$^+$ bordism allows us to also deduce that the $\mathrm{Aut}(\xi)$-orbit of a class $[M]$ in Ω_4^ξ is $\{\pm[M]\}$. We obtain nine equivalence classes: $0, \pm 1, \pm 2, \dots, \pm 7, 8$, detected by the image of the η-invariant in $(\mathbb{Z}/16)/(x \sim -x)$. $\square$

As a consequence of Kreck's classification in Example 5.1.2, we have seen that all unorientable, almost spin 4-manifolds M with $\pi_1(M) \cong G$ are either pin$^+$ or pin$^-$, and that this determines their normal 1-type. This is not true for more general G.

Example 5.3.4. Cappell-Shaneson [**CS71, CS76**] construct a closed, smooth 4-manifold Q that is homeomorphic but not diffeomorphic to $\mathbb{RP}^4$, and show that Q and $\mathbb{RP}^4$ are not stably diffeomorphic. Stolz [**Sto88**] gives another proof of this fact by computing the classes of $\mathbb{RP}^4$ and Q in $\Omega_4^\xi/\mathrm{Aut}(\xi)$. We briefly summarize Stolz' proof.

Since $\pi_1(\mathbb{RP}^4) \cong \mathbb{Z}/2$ and $w_2(\mathbb{RP}^4) = 0$, the proof of Theorem 5.3.2 shows $M\xi \simeq MTPin^+$, $\Omega_4^\xi \cong \mathbb{Z}/16$, and the set of stable diffeomorphism classes is $\Omega_4^\xi/\mathrm{Aut}(\xi) \cong (\mathbb{Z}/16)/(x \sim -x)$. Stolz [**Sto88**] chooses an isomorphism $\Omega_4^\xi \xrightarrow{\cong} \mathbb{Z}/16$ and shows that it sends the two pin$^+$ structures on $\mathbb{RP}^4$ to ± 1 and the two pin$^+$ structures on Q to ± 9; therefore $\mathbb{RP}^4$ and Q are not stably diffeomorphic.

5.3.2. The totally non-spin case.

Theorem 5.3.5. *There are four stable diffeomorphism classes of unorientable, totally non-spin 4-manifolds with $\pi_1(M) \cong G$. The Stiefel-Whitney numbers w_4 and w_2^2 detect these classes.*

This theorem can also be extracted from work of Davis [**Dav05**, Theorem 2.3], who computes a different set of invariants.

PROOF. Example 5.1.2 shows there is only one unorientable normal 1-type in this case: $w_1(\nu) \neq 0$, so it must be pulled back from $p^*x \in H^1(B\pi_1(M);\mathbb{Z}/2)$. Since $p^*x = w_1(p^*\sigma)$, the normal 1-type is

$$V_{\mathrm{SO}} \oplus (p^*\sigma - 1)\colon B\mathrm{SO} \times B\pi_1(M) \longrightarrow B\mathrm{O} \tag{5.3.6}$$

and its Thom spectrum is $MSO \wedge (B\pi_1(M))^{p^*\sigma-1}$, which by Theorem 5.2.1 is equivalent to $MSO \wedge (B\mathbb{Z}/2)^{\sigma-1}$.

Lemma 5.3.7 (Gray [**Gra80**, §2])**.** *There is an equivalence $MSO \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq MO$.*

So $\Omega_4^\xi \cong \Omega_4^{\mathrm{O}}$, and $\Omega_4^{\mathrm{O}} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ [**Tho54**, Corollaire following Théorème IV.12]. The $\mathrm{Aut}(\xi)$-action is trivial. To see this, first observe that $\mathrm{Aut}(\mathrm{id}\colon B\mathrm{O} \to B\mathrm{O})$ is trivial, hence acts trivially on Ω_4^{O}. Thus the $\mathrm{Aut}(\xi)$-orbit of a class in Ω_4^ξ maps to a single class in Ω_4^{O}, so $\mathrm{Aut}(\xi)$-orbits are singletons. Therefore any complete bordism invariant for Ω_4^{O} also is a complete stable diffeomorphism invariant for the normal 1-type ξ, such as (w_2^2, w_4). $\square$

Remark 5.3.8. If M is pin$^+$ or pin$^-$, then its double cover is spin, and hence M is almost spin. So totally non-spin manifolds are neither pin$^+$ nor pin$^-$. Therefore the three normal 1-types that occur when $\pi_1(M) \cong G$ and M is unorientable are the cases pin$^+$, pin$^-$, and neither pin$^+$ nor pin$^-$.

5.4. Stable homeomorphism classes

In order to classify stable homeomorphism classes of topological 4-manifolds, we run the same story, replacing $B\mathrm{O}$ with $B\mathrm{Top}$, where Top_n is the topological group of homeomorphisms $\mathbb{R}^n \to \mathbb{R}^n$ that fix the origin and $\mathrm{Top} \coloneqq \varinjlim_n \mathrm{Top}_n$. As in the previous section, fix a group G finite of order 2 mod 4 with a surjective map $p\colon G \to \mathbb{Z}/2$, and assume that $\mathbb{Z}/2$ acts trivially on $H^*(B\ker(p))$.

Given a topological manifold M, there is a map $\nu\colon M \to B\mathrm{Top}$ called the *stable topological normal bundle*, so we can define normal 1-types, and Kreck's classification argument still applies in the topological setting, this time determining stable homeomorphism classes.

Lemma 5.4.1. *Let M be a closed, unorientable 4-manifold. The possible normal 1-types of M are the same as in Example 5.1.2, except replacing $B\mathrm{O}$ with $B\mathrm{Top}$, $B\mathrm{SO}$ with $B\mathrm{STop}$, and $B\mathrm{Spin}$ with $B\mathrm{TopSpin}$.*

PROOF. The proof is very similar to Kasprowski-Land-Powell-Teichner's determination of the possible normal 1-types of topological 4-manifolds in the orientable case [**KLPT17**, Proposition 4.1]. Since the Stiefel-Whitney classes of a manifold are homotopy invariants, notions of almost spin and totally non-spin make sense for topological manifolds. In the almost-spin case, we have to check that a lift $M \to B\mathrm{TopSpin} \times B\pi_1(M)$ is 2-connected: the proof is the same as in the smooth case, because $\pi_2(B\mathrm{TopSpin}) = 0$. For the totally non-spin case, $\pi_1(B\mathrm{STop}) \cong \mathbb{Z}/2$, detected by w_2, and since M is totally non-spin, $w_2(M) \neq 0$, so the lift is surjective on π_2 just as in the smooth case. $\square$

Our arguments below make use of the fact that bordism groups of topological manifolds are homotopy groups of Thom spectra, which requires a transversality argument. In dimension 4, Scharlemann [**Sch76**] proves the topological transversality theorem that we need. See Teichner [**Tei93**, §IV] for more information.

Let E_8 denote Freedman's E_8 manifold [**Fre82**]. The obstruction to admitting a triangulation defines a bordism invariant $\Omega_4^{\mathrm{Top}} \to \mathbb{Z}/2$ [**KT90b**, §9] which is nonzero on E_8.

5.4.1. The almost spin case. There are topological versions of spin and pin$^\pm$ structures; see Kirby-Taylor [**KT90b**, §9] for details. Kirby-Taylor also produce a homomorphism $S\colon \Omega_4^{\mathrm{TopPin}^+} \to \Omega_2^{\mathrm{TopPin}^-} \cong \Omega_2^{\mathrm{Pin}^-} \cong \mathbb{Z}/8$ sending a pin$^+$ topological 4-manifold M to the pin$^-$ bordism class of a continuously embedded representative of the Poincaré dual of $w_1(M)^2$, which has an induced pin$^-$ structure and a unique smooth structure. Let S' be the invariant sending a topological pin$^+$ 4-manifold M to the image of $S(M)$ in the set $(\mathbb{Z}/8)/(x \sim -x)$.

Theorem 5.4.2.

(1) There are ten stable homeomorphism classes of unorientable pin$^+$ topological 4-manifolds with $\pi_1(M) \cong G$. These classes are detected by the invariant S' constructed above and the triangulation obstruction.

(2) There are two stable homeomorphism classes of unorientable pin$^-$ topological 4-manifolds with $\pi_1(M) \cong G$. These classes are detected by the triangulation obstruction.

Proof. Following the same line of argument as in the proof of Theorem 5.3.2, the two normal 1-types' Thom spectra are $MTopSpin \wedge (B\pi_1(M))^{p^*\sigma-1}$ and $MTopSpin \wedge (B\pi_1(M))^{p^*(3\sigma)-3}$, and Theorem 5.2.1 simplifies these to $MTopSpin \wedge (B\mathbb{Z}/2)^{\sigma-1}$ and $MTopSpin \wedge (B\mathbb{Z}/2)^{3\sigma-3}$, respectively.

Lemma 5.4.3. *There are equivalences $MTopSpin \wedge (B\mathbb{Z}/2)^{\sigma-1} \simeq MTTopPin^-$ and $MTopSpin \wedge (B\mathbb{Z}/2)^{3\sigma-3} \simeq MTTopPin^+$.*

Proof. There are surjective maps $d_n\colon \mathrm{Top}_n \twoheadrightarrow \{\pm 1\}$ given by assigning to a homeomorphism the automorphism it defines on $H_n(\mathbb{R}_n, \mathbb{R}_n \setminus 0) \cong \mathbb{Z}$. These commute with the inclusions $\mathrm{Top}_n \hookrightarrow \mathrm{Top}_{n+1}$, and passing to the colimit defines a map $d\colon \mathrm{Top} \twoheadrightarrow \{\pm 1\}$. This is a topological version of assigning an orthogonal matrix its determinant, classifying whether it preserves or reverses orientation. Given a principal Top-bundle $P \to M$, let $\mathrm{Det}(P) \to M$ be the line bundle $P \times_{\mathrm{Top}} \mathbb{R} \to M$, where Top acts on $\mathbb{R}$ through d. The maps $\mathrm{Top}_n \times \mathrm{O}_1 \to \mathrm{Top}_n \times \mathrm{Top}_1 \to \mathrm{Top}_{n+1}$ allow us to make sense of "$P \oplus n\,\mathrm{Det}(P)$" as a principal Top-bundle.

We abuse notation for a moment to say that a G-structure on a principal Top-bundle $P \to M$ is a reduction of structure group of P from Top to G. Then, just as in the smooth case, there is a natural equivalence between the set of TopPin^--structures on P and the set of TopSpin structures on $P \oplus \mathrm{Det}(P)$, and similarly between the set of TopPin^+-structures on P and the set of TopSpin structures on $P \oplus 3\,\mathrm{Det}(P)$. The proof is the same as in the smooth case. These equivalences are the only facts we need to know about $\mathrm{Pin}^\pm$ in order to prove (4.2.10a) and (4.2.10b) splitting $\mathit{MTPin}^\pm$, so the argument in the topological setting can proceed in the same way. □

Therefore by Theorem 5.2.1, our two normal 1-types are equivalent to $\mathit{MTTopPin}^\pm$. The caveat about switching between pin^+ and pin^- when one passes between the tangent and normal bundles still applies here.

Theorem 5.4.4 (Kirby-Taylor [**KT90b**, Theorem 9.2])**.**

(1) $\Omega_4^{\mathrm{TopPin}^-} \cong \mathbb{Z}/2$*, generated by* E_8*.*

(2) $\Omega_4^{\mathrm{TopPin}^+} \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$*, with* $\mathbb{RP}^4$ *generating the* $\mathbb{Z}/8$ *summand and* E_8 *generating the* $\mathbb{Z}/2$ *summand.*

(3) The map $\Omega_4^{\mathrm{Pin}^+} \to \Omega_4^{\mathrm{TopPin}^+}$ *is identified with a map* $\mathbb{Z}/16 \to \mathbb{Z}/8 \oplus \mathbb{Z}/2$ *which surjects onto the first factor and does not hit* E_8*.*

(4) The homomorphism $S\colon \Omega_4^{\mathrm{TopPin}^+} \to \Omega_2^{\mathrm{TopPin}^-} \cong \Omega_2^{\mathrm{Pin}-} \cong \mathbb{Z}/8$ *sends* $\mathbb{RP}^4$ *to a generator.*

Since $\mathbb{Z}/2$ is rigid, we conclude there are two stable homeomorphism classes in the pin^- case, detected by the triangulation obstruction. For the pin^+ case, the same line of reasoning in the proof of Theorem 5.3.2 allows us to reduce to the case when ξ is a topological pin^+ structure, so we can compute the action of $\mathrm{Aut}(\xi)$ on the generators. Since E_8 is simply connected, it admits a unique topological pin^+ structure, so is fixed by $\mathrm{Aut}(\xi)$. For $\mathbb{RP}^4$, every topological pin^+ structure on arises from a smooth pin^+ structure, so we can reuse our work from Theorem 5.3.2 to conclude the $\mathrm{Aut}(\xi)$-orbit of $\mathbb{RP}^4$ is again $\pm[\mathbb{RP}^4]$. Therefore the set of stable diffeomorphism classes is $((\mathbb{Z}/8)/(x \sim -x)) \times \mathbb{Z}/2$, which has ten elements, and the triangulation obstruction and S' are together a complete invariant. □

5.4.2. The totally non-spin case. By Lemma 5.4.1, there is only one normal 1-type to worry about.

Theorem 5.4.5. *There are eight stable homeomorphism classes of unorientable, totally non-spin topological 4-manifolds with* $\pi_1(M) \cong G$*. The triangulation obstruction and the Stiefel-Whitney numbers* w_4 *and* w_2^2 *are together a complete stable homeomorphism invariant.*

Again, this can be extracted from a theorem of Davis [**Dav05**, Theorem 2.3], who uses a different but equivalent set of invariants.

Proof. Following the same line of reasoning as in Theorem 5.3.5, Lemma 5.4.1 tells us we only have one normal 1-type, and its Thom spectrum is $MSTop \wedge (B\mathbb{Z}/2)^{\sigma-1}$.

Lemma 5.4.6. *There is an equivalence $MTop \simeq MSTop \wedge (B\mathbb{Z}/2)^{\sigma-1}$.*

Proof. The proof goes through as in the smooth case, since we have a determinant map and the fact that for any Top-bundle $P \to M$, $P \oplus \mathrm{Det}(P)$ is canonically oriented, analogously to the smooth case. $\square$

So we need to calculate Ω_4^{Top}.

Proposition 5.4.7. *$\Omega_4^{\mathrm{Top}} \cong (\mathbb{Z}/2)^{\oplus 3}$, with a basis given by the classes of $\mathbb{RP}^4$, $\mathbb{RP}^2 \times \mathbb{RP}^2$, and E_8. The Stiefel-Whitney numbers w_4 and w_2^2 and the triangulation obstruction are linearly independent on this bordism group.*

Proof. Draw the Atiyah-Hirzebruch spectral sequence computing Ω_4^{Top} as $\Omega_4^{\mathrm{STop}}((B\mathbb{Z}/2)^{\sigma-1})$. It collapses for degree reasons in total degree 4 and below, and the 4-line of the E_∞-page has order 8. Therefore it suffices to find three linearly independent nonzero elements of Ω_4^{Top}, which can be done by computing w_4, w_2^2, and the triangulation obstruction on $\mathbb{RP}^4$, $\mathbb{RP}^2 \times \mathbb{RP}^2$, and E_8. $\square$

Just as in the smooth case, $\mathrm{Aut}(\xi)$ acts trivially. $\square$

Remark 5.3.8 also applies in the topological case: the three normal 1-types for unorientable topological manifolds with $\pi_1(M) \cong G$ are precisely the cases where M has a topological pin$^+$ structure, M has a topological pin$^-$ structure, and M has neither.

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